Homotopy Type Theory

Equality as Equality

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Introduction

In HoTT there are two notions of equality:

- *judgemental equality* $\Gamma \vdash a \equiv b : T$
- *propositional equality* $a =_T b$

The former models *conversion*

The latter is

- the *identity type* over the type T
- the *space of paths* from a to b in the space T

Are these really equalities?

Equivalence relation

Equality is an *equivalence relation*

MLTT, on which HoTT is based on, contains

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A}\ \equiv\text{-refl} \qquad \frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A}\ \equiv\text{-sym}$$

$$\frac{\Gamma \vdash a \equiv b : A \quad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}\ \equiv\text{-trans}$$

Hence, judgemental equality is an equivalence relation…

…between regular judgements!

Equivalence relation

The identity type is defined by

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash = : \Pi A : \mathscr{U}_i, a : A, b : A. \mathscr{U}_i} \ {=}\text{-form}$$

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathsf{refl} : \Pi A : \mathscr{U}_i, a : A. a =_A a} \ {=}\text{-intro}$$

$$\frac{\Gamma\ \mathsf{ctx}}{\begin{array}{l}\Gamma \vdash \mathsf{ind}_= : \Pi A : \mathscr{U}_i. \\ \qquad\qquad \Pi P : (\Pi x : A, y : A. x =_A y \to \mathscr{U}_i). \\ \qquad\qquad \Pi c_1 : (\Pi x : A. P\, x\, x\, (\mathsf{refl}\, A\, x)). \\ \qquad\qquad \Pi a : A, b : A. \Pi e : a =_A b. P\, a\, b\, e\end{array}} \ {=}\text{-elim}$$

$$\frac{\begin{array}{cc}\Gamma \vdash A : \mathscr{U}_i & \Gamma \vdash P : \Pi x : A, y : A. x =_A y \to \mathscr{U}_i \\ \Gamma \vdash a : A & \Gamma \vdash c_1 : \Pi x : A. P\, x\, x\, (\mathsf{refl}\, A\, x)\end{array}}{\Gamma \vdash \mathsf{ind}_=\, A\, P\, c_1\, a\, a\, (\mathsf{refl}\, A\, a) \equiv c_1\, a : P\, a\, a\, (\mathsf{refl}\, A\, a)} \ {=}\text{-comp}$$

Equivalence relation

Lemma 2.1 (Path inversion)

For every type A *and every* $x, y : A$ *there is a function* $(\_)^{-1} : x =_A y \to y =_A x$ *such that* $\mathsf{refl}_x^{-1} \equiv \mathsf{refl}_x$*.*

Proof.

Define $P :\equiv \lambda x : A, y : A, p : x =_A y.\, y =_A x$ and $c_1 :\equiv \lambda x : A.\,\mathsf{refl}_x$.
Thus, $(\_)^{-1} :\equiv \mathsf{ind}_= \, A\, P\, c_1\, x\, y$.
Hence $\mathsf{refl}_x^{-1} \equiv \mathsf{ind}_= \, A\, P\, c_1\, x\, x\, \mathsf{refl}_x \equiv c_1\, x \equiv \mathsf{refl}_x$. □

Generally, type checking is implicit in proofs

Equivalence relation

Lemma 2.2 (Path composition)

For every type A and every $x, y, z : A$ there is a function

$$\_\cdot\_ : x = y \to y = z \to x = z$$

such that $\mathsf{refl}_x \cdot \mathsf{refl}_x \equiv \mathsf{refl}_x$ *for any* $x : A$.

Proof.

$$D :\equiv \lambda x : A, y : A, p : x =_A y. \Pi z : A, q : y =_A z. x =_A z$$

$$E :\equiv \lambda x : A, z : A, q : x =_A z. x =_A z$$

$$e :\equiv \lambda x : A. \mathsf{refl}_x$$

$$d :\equiv \lambda x : A. \mathsf{ind}_= A\, E\, e$$

$$f :\equiv \lambda A : \mathscr{U}_i, x : A, y : A, z : A, p : x =_A y, q : y =_A z. \mathsf{ind}_= A\, D\, d\, x\, y\, p\, z\, q$$

Then $\cdot :\equiv f\, A\, x\, y\, z$ is the sought term.
The judgemental equality $\mathsf{refl}_x \cdot \mathsf{refl}_x \equiv \mathsf{refl}_x$ follows by $=$–comp. □

Equivalence relation

Equality is an *equivalence relation*

By =–intro, $=_A$ is reflexive

Lemma 2.1 tells that $=_A$ is symmetric

Lemma 2.2 tells that $=_A$ is transitive

Hence $=_A$ is an equivalence relation between terms in the same type A

Congruence

Equality is a *congruence*

It suffices to show that equality is a congruence w.r.t. the structural constructors: application, abstraction, function space formation:

1. if $f = g$ then $f\,a = g\,a$
2. if $a = b$ then $f\,a = f\,b$
3. if $A = B$ then $(\lambda x : A.\, e) = (\lambda x : B.\, e)$
4. if $a = b$ then $(\lambda x : A.\, a) = (\lambda x : A.\, b)$
5. if $A = B$ then $(\Pi x : A.\, C) = (\Pi x : B.\, C)$
6. if $B = C$ then $(\Pi x : A.\, B) = (\Pi x : A.\, C)$

Because of the way in which we defined inductive types, it follows that equality is a congruence also w.r.t. them if it is w.r.t. the fundamental constructors above.

Congruence

Equality is a *congruence*

MLTT, on which HoTT is based, contains

$$\frac{\Gamma \vdash A \equiv A' : \mathscr{U}_i \quad \Gamma, x : A \vdash B \equiv B' : \mathscr{U}_i}{\Gamma \vdash \Pi x : A. B \equiv \Pi x : A'. B' : \mathscr{U}_i}\ \Pi\text{−form−eq}$$

$$\frac{\Gamma, x : A \vdash b \equiv b' : B \quad \Gamma \vdash A \equiv A' : \mathscr{U}_i}{\Gamma \vdash \lambda x : A. b \equiv \lambda x : A'. b' : \Pi x : A. B}\ \Pi\text{−intro−eq}$$

$$\frac{\Gamma \vdash f \equiv g : \Pi x : A. B \quad \Gamma \vdash a \equiv a' : A}{\Gamma \vdash f\, a \equiv g\, a' : B[a/x]}\ \Pi\text{−elim−eq}$$

Hence judgemental equality is a congruence.

Lemma 3.1 (Transport)

Let $P : A \to \mathcal{U}_i$ *and* $p : x =_A y$. *Then there is* $p_* : P\,x \to P\,y$.

Proof.
Pose

$$D :\equiv \lambda x : A, y : A, p : x = y.\, P\,x \to P\,y$$
$$d :\equiv \lambda x : A.\, \mathsf{id}_{(P\,x)}$$

Thus $p_* :\equiv \mathsf{ind}_{=}\, A\, D\, d\, x\, y\, p$. □

Lemma 3.2

If $f =_{\Pi x:A.B} g$ *then* $f\,a =_{B[a/x]} g\,a$ *for every* $a : A$*.*

Proof.

Let $P :\equiv \lambda h : (\Pi x : A. B). f\,a =_{B[a/x]} h\,a$. Let $p : f = g$.
Then $p_* : f\,a =_{B[a/x]} f\,a \to f\,a =_{B[a/x]} g\,a$ by Lemma 3.1.
Thus $p_* (\mathsf{refl}\, B[a/x]\, (f\,a)) : f\,a =_{B[a/x]} g\,a$. □

Congruence

Apparently, it would be simple to prove also

Lemma 3.3

If $a =_A b$ then $f\,a =_{B[a/x]} f\,b$ for every $f : \Pi x : A.\, B$.

Proof.

Let $P :\equiv \lambda y : A.\, f\,a =_{B[a/x]} f\,y$. Let $p : a = b$.
Then $p_* : f\,a =_{B[a/x]} f\,a \to f\,a =_{B[a/x]} f\,b$ by Lemma 3.1.
Thus $p_*\,(\mathrm{refl}\,B[a/x]\,(f\,a)) : f\,a =_{B[a/x]} f\,b$. □

The part in red is **wrong** because P does not type check.
Observe how the proof is correct when $f : A \to B$.

Function extensionality

Definition 3.4 (Homotopy)

Let $f, g : \Pi x : A. B$. Then a *homotopy* from f to g is a function of type

$$(f \sim g) :\equiv \Pi x : A. f\, x =_B g\, x \quad .$$

Lemma 3.5

Let $f, g : \Pi x : A. B$. *Then there is a term* happly *such that*

$$\mathsf{happly} : (f = g) \to (f \sim g) \quad .$$

Proof.

$$D :\equiv \lambda y, z : (\Pi x : A. B), q : y = z. y \sim z$$
$$d :\equiv \lambda y : (\Pi x : A. B). \lambda x : A. \mathsf{refl}_{y\, x}$$

Then $\mathsf{happly} :\equiv \mathsf{ind}_{=}\, (\Pi x : A. B)\, D\, d\, f\, g$ has type $(f = g) \to (f \sim g)$. □

Function extensionality

Theorem 3.6

Let $f, g : \Pi x : A.B$*. Then there is a term* funext *such that*

$$\mathrm{funext} : (f \sim g) \to (f = g) \ .$$

Proof.

By a complex proof due to Voevodsky. □

Beware that the proof is not clean: it uses many preliminary results, some of which may have been proved using function extensionality!
However, it is believed that it can be performed avoiding that principle.

Congruence

Lemma 3.7

If $\Pi x : A.\, a =_B b$ *then* $(\lambda x : A.\, a) =_{\Pi x : A.\, B} (\lambda x : A.\, b)$.

Proof.

Observe that $(\lambda x : A.\, a)\, x \equiv a$ and $(\lambda x : A.\, b)\, x \equiv b$, thus the hypothesis becomes $\Pi x : A.\, (\lambda x : A.\, a)\, x =_B (\lambda x : A.\, b)\, x$, that is, $(\lambda x : A.\, a) \sim (\lambda x : A.\, b)$. The conclusion follows by Theorem 3.6. □

Congruence

However, as before, the following, apparently natural proof, is wrong:

Lemma 3.8

If $A =_{\mathscr{U}_i} B$ *and* $\Gamma, x : A \vdash a : C$ *then* $(\lambda x : A.\, a) =_{\Pi x : A.\, C} (\lambda x : B.\, a)$.

Proof.

Let $P :\equiv \lambda z : \mathscr{U}_i.\, (\lambda x : A.\, a) =_{\Pi x : A.\, C} (\lambda x : z.\, a)$. Let $p : A =_{\mathscr{U}_i} B$.
Then $p_* : (\lambda x : A.\, a) =_{\Pi x : A.\, C} (\lambda x : A.\, a) \to (\lambda x : A.\, a) =_{\Pi x : A.\, C} (\lambda x : B.\, a)$ by Lemma 3.1.
Thus $p_* (\mathsf{refl}\, (\Pi x : A.\, C)\, (\lambda x : A.\, a)) : (\lambda x : A.\, a) =_{\Pi x : A.\, C} (\lambda x : B.\, a)$. □

Congruence

Lemma 3.9

If $A =_{\mathscr{U}_i} B$ *and* $\Gamma, x : A \vdash C : \mathscr{U}_i$ *then* $(\Pi x : A. C) =_{\mathscr{U}_i} (\Pi x : B. C)$.

Proof.

Let $P :\equiv \lambda z : \mathscr{U}_i. (\Pi x : A. C) =_{\mathscr{U}_i} (\Pi x : z. C)$. Let $p : A =_{\mathscr{U}_i} B$.
Then $p_* : (\Pi x : A. C) =_{\mathscr{U}_i} (\Pi x : A. C) \to (\Pi x : A. C) =_{\mathscr{U}_i} (\Pi x : B. C)$ by Lemma 3.1.
Thus $p_* (\mathsf{refl}_{\mathscr{U}_i} (\Pi x : A. C)) : (\Pi x : A. C) =_{\mathscr{U}_i} (\Pi x : B. C)$. □

Congruence

Lemma 3.10

If $\Pi x : A. (B =_{\mathscr{U}_i} C)$ *then* $(\Pi x : A. B) =_{\mathscr{U}_i} (\Pi x : A. C)$.

Proof.
Since $B \equiv (\lambda x : A. B)\, x$ and $C \equiv (\lambda x : A. C)\, x$, the hypothesis becomes $(\lambda x : A. B) \sim (\lambda x : A. C)$.
Then, by Theorem 3.6, there is $t : (\lambda x : A. B) =_{A \to \mathscr{U}_i} (\lambda x : A. C)$. Pose

$$D :\equiv \lambda y, z : (A \to \mathscr{U}_i), q : y =_{(A \to \mathscr{U}_i)} z. (\Pi x : A. y\, x) =_{\mathscr{U}_i} (\Pi x : A. z\, x)$$
$$d :\equiv \lambda y : (A \to \mathscr{U}_i). \mathsf{refl}_{(\Pi x : A. y\, x)}$$

Observe how $d : D\, y\, y\, \mathsf{refl}_y$. Thus

$$\mathsf{ind}_{=} (A \to \mathscr{U}_i)\, D\, d\, (\lambda x : A. B)\, (\lambda x : A. C)\, t$$

inhabits

$$(\Pi x : A. B) =_{\mathscr{U}_i} (\Pi x : A. C) \ . \qquad \square$$

Congruence

Equality is a *congruence*

By Lemmas 3.9, and 3.10, equality $=_T$ is a congruence w.r.t. function space formation for every type T.

By Lemmas 3.2 and 3.3, equality $=_T$ is a congruence w.r.t. application **when dealing with non-dependent functions**.

By Lemma 3.7, equality $=_T$ is a congruence w.r.t. abstraction **limited to the body**, i.e., having a fixed domain.

Substitution

The proofs of Lemmas 3.7 and 3.10 are correct, but they rely on a principle we have not shown: if $a \equiv b$ then $a =_T b$.

Also, we have shown that judgemental equality is a congruence, but we omitted to consider types:

- under the hypotheses of Π–intro–eq, we should prove **both** $\Gamma \vdash \lambda x : A. b \equiv \lambda x : A'. b' : \Pi x : A. B$ and $\Gamma \vdash \lambda x : A. b \equiv \lambda x : A'. b' : \Pi x : A'. B$ **because** $\Pi x : A. B \equiv \Pi x : A'. B$.
- under the hypotheses of Π–elim–eq, we should prove **both** $\Gamma \vdash f\, a \equiv g\, a' : B[a/x]$ and $\Gamma \vdash f\, a \equiv g\, a' : B[a'/x]$ **because** $B[a/x] \equiv B[a'/x]$.

More in general, we would expect that if $a \equiv b$ then $C[a] \equiv C[b]$.

And we have not yet discussed **why** Lemmas 3.3 and 3.8 fail.

Substitution

Lemma 4.1

If $\Gamma \vdash a \equiv b : A$ *then* $\Gamma \vdash \mathrm{refl}_a : a =_A b$*.*

Proof.

Observe that $(a =_A a) \equiv_L (a =_A x)[a/x]$ and $(a =_A b) \equiv_L (a =_A x)[b/x]$, where $\equiv_L$ means syntactically equal.
Thus $\Gamma \vdash (a =_A a) \equiv (a =_A b) : \mathscr{U}_i$ because $\equiv$ is congruence.
From the hypothesis, we get $\Gamma \vdash a : A$ and $\Gamma \vdash A : \mathscr{U}_i$ by Lemmas 4.2 and 4.3.
Hence $\Gamma \vdash \mathrm{refl}_a : a =_A a$ by $=$–intro.
Thus $\Gamma \vdash \mathrm{refl}_a : a =_A b$ by $\equiv$–subst. □

From $\Gamma \vdash \Pi x : A.a =_B b : \mathscr{U}_i$ we can derive $\Gamma, x : A \vdash a =_B b$, and in turn $\Gamma, x : A \vdash a : B$ by Lemma 4.2. Then the missing part in Lemmas 3.7 and 3.10 is immediately obtained by Π–comp and Lemma 4.1.

Inversion

Lemma 4.2 (Inversion)

1. *If* $\Gamma \vdash x : T$ *with* x *a variable, then* $\Gamma \vdash x : A$ *by* Vble *and* $\Gamma \vdash A \equiv T : \mathcal{U}_i$*, or* $\Gamma \vdash A \equiv \mathcal{U}_i : \mathcal{U}_k$ *and* $\Gamma \vdash T \equiv \mathcal{U}_j : \mathcal{U}_k$, $i < j < k$.
2. *If* $\Gamma \vdash \kappa : T$ *with* κ *a constant, then* $\Gamma \vdash \kappa : A$ *by the corresponding introduction rule, and* $\Gamma \vdash A \equiv T : \mathcal{U}_i$*, or* $\Gamma \vdash A \equiv \mathcal{U}_i : \mathcal{U}_k$ *and* $\Gamma \vdash T \equiv \mathcal{U}_j : \mathcal{U}_k$, $i < j < k$.
3. *If* $\Gamma \vdash \mathcal{U}_i : T$ *then* $\Gamma \vdash \mathcal{U}_i : \mathcal{U}_{i+1}$ *by* $\mathcal{U}$–intro *and* $\Gamma \vdash T \equiv \mathcal{U}_j$, $i < j$.
4. *If* $\Gamma \vdash \Pi x : A. B : T$ *then* $\Gamma \vdash \Pi x : A. B : \mathcal{U}_i$ *by* Π–form*, and* $\Gamma \vdash T \equiv \mathcal{U}_j$, $i \leq j$.
5. *If* $\Gamma \vdash \lambda x : A. b : T$ *then* $\Gamma \vdash \lambda x : A. b : \Pi x : A. B$ *by* Π–intro*, and* $\Gamma \vdash \Pi x : A. B \equiv T : \mathcal{U}_i$.
6. *If* $\Gamma \vdash f\, a : T$ *then* $\Gamma \vdash f\, a : A$ *by* Π–elim*, and* $\Gamma \vdash A \equiv T : \mathcal{U}_i$*, or* $\Gamma \vdash A \equiv \mathcal{U}_i : \mathcal{U}_k$ *and* $\Gamma \vdash T \equiv \mathcal{U}_j : \mathcal{U}_k$, $i < j < k$.

Proof.

It suffices to observe that the last step of a derivation of $\Gamma \vdash t : T$ is either an instance of $\mathcal{U}$–cumul or $\equiv$–subst from a premise $\Gamma \vdash t : T'$, or an instance of a rule introducing the main operator. □

Pieces apart

Lemma 4.3

1. *If* $\Gamma, x : A, \Delta \,\mathrm{ctx}$ *then* $\Gamma \,\mathrm{ctx}$, *and* $\Gamma \vdash A : \mathscr{U}_i$.
2. *If* $\Gamma \vdash a : A$ *then* $\Gamma \,\mathrm{ctx}$, *and* $\Gamma \vdash A : \mathscr{U}_i$.
3. *If* $\Gamma \vdash a \equiv b : A$ *then* $\Gamma \,\mathrm{ctx}$, $\Gamma \vdash A : \mathscr{U}_i$, *and* $\Gamma \vdash a : A$, $\Gamma \vdash b : A$.

Proof.

By a long deep induction on the derivations. □

Substitution

Theorem 4.4 (Substitution)

If $\Gamma \vdash a \equiv b : A$ *and* $\Gamma, x : A \vdash C : B$ *then* $\Gamma \vdash B[a/x] \equiv B[b/x] : \mathscr{U}_i$ *and* $\Gamma \vdash C[a/x] \equiv C[b/x] : B[a/x]$.

Proof.

From $\Gamma, x : A \vdash C : B$, we get $\Gamma, x : A \vdash B : \mathscr{U}_i$, and from $\Gamma \vdash a \equiv b : A$ we get $\Gamma \vdash a : A$ and $\Gamma \vdash b : A$ by Lemma 4.3.

Then $\Gamma \vdash \lambda x : A. B \equiv \lambda x : A. B : A \to \mathscr{U}_i$ by $\equiv$−refl after Π−intro, so
$\Gamma \vdash (\lambda x : A. B)\, a \equiv (\lambda x : A. B)\, b : \mathscr{U}_i$ by Π−elim−eq.
But $\Gamma \vdash B[a/x] \equiv (\lambda x : A. B)\, a : \mathscr{U}_i$ by $\equiv$−sym after Π−comp.
Also $\Gamma \vdash (\lambda x : A. B)\, b \equiv B[b/x] : \mathscr{U}_i$ by Π−comp.
Hence $\Gamma \vdash B[a/x] \equiv B[b/x] : \mathscr{U}_i$ by $\equiv$−trans.

Similarly, $\Gamma \vdash (\lambda x : A. C)\, a \equiv (\lambda x : A. C)\, b : B[a/x]$, thus
$\Gamma \vdash C[a/x] \equiv C[b/x] : B[a/x]$ following the same reasoning. □

Equivalence vs Equality

Hence we solved all the previously listed issues about judgemental equality.
It remains to analyse why Lemmas 3.3 and 3.8 fail.

Using some black magic, one can prove

Lemma 4.5

If $\Gamma \vdash \Pi x : A. B \equiv \Pi x : A'. B' : \mathscr{U}_i$ *then* $\Gamma \vdash A \equiv A' : \mathscr{U}_i$ *and* $\Gamma, x : A \vdash B \equiv B' : \mathscr{U}_i$.

Equivalence vs Equality

Suppose Lemma 3.8 holds: if $A =_{\mathscr{U}_i} B$ and $\Gamma, x : A \vdash a : C$ then $(\lambda x : A.\, a) =_{\Pi x : A.\, C} (\lambda x : B.\, a)$.

Pose $a :\equiv x$ and $C :\equiv A$ and assume $A =_{\mathscr{U}_i} B$.
Then $\mathsf{id}_A =_{\Pi x : A.\, A} \mathsf{id}_B$, so $\Gamma \vdash \mathsf{id}_B : \Pi x : A.\, A$ by Lemma 4.3, thus $\Gamma, x : B \vdash x : B$ and $\Gamma \vdash \Pi x : A.\, A \equiv \Pi x : B.\, B : \mathscr{U}_i$ by Lemma 4.2.
Hence $\Gamma \vdash A \equiv B : \mathscr{U}_i$ by Lemma 4.5.

However, for example, $\mathbf{1} + \mathbf{1} = \mathbf{2}$ but $\mathbf{1} + \mathbf{1} \not\equiv \mathbf{2}$. Thus Lemma 3.8 is untenable.

Equivalence vs Equality

Consider Lemma 3.3: if $a =_A b$ then $f\,a =_{B[a/x]} f\,b$ for every $f : \Pi x : A. B$.
It holds if $f : A \to B$, so assume f to be dependent.
And it is significant when $a \not\equiv b$, so assume this fact.
Under these constraints, assume Lemma 3.3.

Observe that if $t : C$ and $t : D$ then either $C \equiv D$, or both $C \equiv \mathcal{U}_i$ and $D \equiv \mathcal{U}_j$, $i \neq j$ by Lemma 4.2.

Take $f :\equiv \lambda x : A. \mathsf{refl}_x : \Pi x : A. x =_A x$. Let $a =_A b$.

Then either $(a =_A a) \equiv (b =_A b)$ or $(a =_A a)$ and $(b =_A b)$ are judgementally equivalent to distinct universes. The latter case is impossible since Church-Rosser Theorem holds in the pure calculus, and $\equiv$ implies convertibility.

Thus, in the pure calculus a converts to b, that is (*) $a \equiv b$, contradiction.
Therefore, Lemma 3.3 in its full generality is untenable.

(*) This is a consequence of Church-Rosser Theorem in MLTT, which has been *almost* proved by the speaker.

Congruence, again

A notion of equality is a *congruence* w.r.t. path operations when

- if $p, q : a =_A b$ and $p =_{a =_A b} q$ then $p^{-1} =_{b =_A a} q^{-1}$
- if $p, q : a =_A b$, $s : c =_A a$ and $p =_{a =_A b} q$ then $s \cdot p =_{c =_A b} s \cdot q$
- if $p, q : a =_A b$, $s : b =_A c$ and $p =_{a =_A b} q$ then $p \cdot s =_{a =_A c} q \cdot s$

Clearly, by Lemma 4.1, this notion makes sense for propositional equality only.

Congruence, again

Proposition 4.6

If $p, q : a =_A b$ *and* $p =_{a=_A b} q$ *then*

1. $p^{-1} =_{b=_A a} q^{-1}$
2. *if* $s : c =_A a$ *then* $s \centerdot p =_{c=_A b} s \centerdot q$
3. *if* $s : b =_A c$ *then* $p \centerdot s =_{a=_A c} q \centerdot s$

Proof.

Let $h : p = q$. Pose

$$
\begin{aligned}
D_1 &:\equiv \lambda x, y : a =_A b.\, x = y \to x^{-1} = y^{-1} \\
D_2 &:\equiv \lambda x, y : a =_A b.\, x = y \to s \centerdot x = s \centerdot y \\
D_3 &:\equiv \lambda x, y : a =_A b.\, x = y \to x \centerdot s = y \centerdot s \\
d_1 &:\equiv \lambda x : a =_A b.\, \mathsf{refl}_{x^{-1}} \\
d_2 &:\equiv \lambda x : a =_A b.\, \mathsf{refl}_{s \centerdot x} \\
d_3 &:\equiv \lambda x : a =_A b.\, \mathsf{refl}_{x \centerdot s}
\end{aligned}
$$

then $\mathrm{ind}(a =_A b)\, D_i\, d_i\, p\, q\, h$ for $i = 1, 2, 3$ inhabits the various cases. □

Conclusion

The good news:

- judgemental and propositional equalities are equivalence relations
- judgemental equality is a congruence w.r.t. type formers
- judgemental equality, thanks to Lemma 4.1 and Proposition 4.6, is a (trivial) congruence w.r.t. path operations
- propositional equality is a congruence w.r.t. path operations

Conclusion

The bad news: in general, propositional equality is **not** a congruence w.r.t. type formers.

However, it is **to some extent**.

Hence, reasoning with equivalences ($\simeq$) is unavoidable in HoTT.

The moral is that there is a small but significant disagreement between the H (homotopy) and the double T (type theory) in HoTT, that prevents a natural encapsulation of the homotopical reasoning in the logical/type theoretical one.

Conclusion

Finally, structural proof-theoretic properties of MLTT play a fundamental role in the fine analysis of HoTT, see, e.g., Church-Rosser Theorem.

This aspect of HoTT is poorly developed.

More in general, HoTT, in its current stage of development, lacks a clean, systematic presentation which is, in the author's opinion, the biggest obstacle to its study and use.

References

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The end

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