MATHEMATICAL LOGIC — ASSIGNMENT THREE

(1) Let α be an ordinal. Prove that $\alpha 1 = \alpha$.

By definition $\alpha 1$ is the unique ordinal such that there is $h\colon S \to \alpha 1$ bijective and monotone where $S = \langle \bigsqcup_{i \in 1} \alpha; \leq \rangle$ with $x \leq y$ when either $x \in \alpha_i$, $y \in \alpha_j$ and $i < j$, or $x, y \in \alpha_i$ for some $i \in 1$ and $x \leq y$ in α.

Since $1 = \{\emptyset\}$, $\bigsqcup_{i \in 1} \alpha = \{(x, \emptyset)\colon x \in \alpha\} \cong \alpha$ by $h\colon S \to \alpha$ defined as $(x, \emptyset) \mapsto x$. Clearly, h is bijective and monotone since only the second case in the definition of $\leq$ on S applies. Hence, $\alpha 1 = \alpha$ by definition.

(2) State and prove Kleene's fixed point theorem.

See Theorem 16.23 in the slides.

(3) Prove that the Axiom of Choice on a finite family is derivable in ZF: every finite collection $\{X_i\}_{1 \leq i \leq n \in \mathbb{N}}$ of non-empty sets has a choice function $f\colon \{X_i\}_{1 \leq i \leq n} \to \bigcup_{1 \leq i \leq n} X_i$ such that $f(X_i) \in X_i$ for every $1 \leq i \leq n$.

Observe that a non-empty set A contains at least one element: this follows because the empty set is unique by extensionality.

Then, proceed by induction on n:

- if $n = 0$, the function $f\colon \emptyset \to \emptyset$ maps no element in no element;
- if $n = k + 1$, there is a choice function $g\colon \{X_i\}_{1 \leq i \leq k} \to \bigcup_{1 \leq i \leq k} X_i$ such that $f(X_i) \in X_i$ for every $1 \leq i \leq k$ by induction hypothesis. Pose $f(X_i) = g(X_i)$ for every $1 \leq i \leq k$, and $f(X_n) = a$ with $a \in X_n$, which has to exist since X_n is non empty.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: May 23$^{\text{th}}$, 2022.