MATHEMATICAL LOGIC — ASSIGNMENT TWO

(1) Prove $\vdash (\forall x.\, A \supset B) = ((\exists x.\, A) \supset B)$ when $x \notin \mathsf{FV}(B)$. Show a counterexample when $x \in \mathsf{FV}(B)$.

$$\dfrac{\dfrac{\dfrac{[\exists x.\, A]^1 \qquad \dfrac{[A]^2 \qquad \dfrac{[\forall x.\, A \supset B]^3}{A \supset B}\,{\scriptstyle \forall\mathrm{E}}}{B}\,{\scriptstyle \supset\mathrm{E}}}{B}\,{\scriptstyle \exists\mathrm{E}^2}}{(\exists x.\, A) \supset B}\,{\scriptstyle \supset\mathrm{I}^1}}{(\forall x.\, A \supset B) \supset ((\exists x.\, A) \supset B)}\,{\scriptstyle \supset\mathrm{I}^3} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^1}{\exists x.\, A}\,{\scriptstyle \exists\mathrm{I}} \qquad [(\exists x.\, A) \supset B]^2}{B}\,{\scriptstyle \supset\mathrm{E}}}{A \supset B}\,{\scriptstyle \supset\mathrm{I}^1}}{\forall x.\, A \supset B}\,{\scriptstyle \forall\mathrm{I}}}{((\exists x.\, A) \supset B) \supset (\forall x.\, A \supset B)}\,{\scriptstyle \supset\mathrm{I}^2}$$

Consider the model of arithmetic in which x is interpreted in 1. Let A be the formula saying x is even and B the formula saying $x+1$ is odd. Then $\forall x.\, A \supset B$ is true, but $(\exists x.\, A) \supset B$ is false.

(2) State the Downward Löwenheim-Skolem Theorem.

This is Theorem 12.5 in the slides.

(3) Show that there is an alternative model of the rational numbers in which $\sqrt{2}$ is rational (Hint: use the decimal expansion of $\sqrt{2}$).

Take the language of the theory $\mathbb{Q}$ of rational numbers and extend it with the constant q. Consider the following axioms:

$$\phi_k \equiv 1.d_1 \cdots d_k \leq q \wedge q \leq 1.d_1 \cdots d_k + 10^{-k}$$

for every $k \in \mathbb{N}$ with $1.d_1 \cdots$ the infinite decimal expansion of $\sqrt{2}$ as a real number. Call $\Phi = \{\phi_k : k \in \mathbb{N}\}$.

Let Ξ be any finite subset of $\mathbb{Q} \cup \Phi$.

If $\Xi \cap \Phi = \emptyset$, then Ξ has the rational numbers as a model in which q is interpreted in, let say, 1.

Otherwise $\Xi \cap \Phi \neq \emptyset$ and it is finite. Hence there is m such that $\phi_m \in \Xi$ and $\phi_i \notin \Xi$ for every $i > m$. Then Ξ has the rational numbers as a model posing $q = 1.d_1 \cdots d_m$.

Therefore, $\mathbb{Q} \cup \Phi$ has a model $\mathcal{M}$ by the Compactness Theorem. In particular $\mathcal{M}$ is a model for $\mathbb{Q}$.

In this model $(1.d_1 \cdots d_n)^2 \leq q^2 \leq (1.d_1 \cdots d_n + 10^{-n})^2$ for every $n \in \mathbb{N}$. However, $(1.d_1 \cdots d_n)^2 \leq 2 \leq (1.d_1 \cdots d_n + 10^{-n})^2$ for every $n \in \mathbb{N}$ by definition, thus q^2 cannot be distinguished from 2 inside the model (but it may still be different by an infinitesimal quantity).

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: May 2^{nd} 2022.