Mathematical Logic

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Constructive Mathematics

- 16/01/2017, third exercise
- 25/05/2018, third exercise
- 10/01/2019, third exercise
- 10/01/2020, first exercise

16/01/2017 (3)

Show that $\phi \equiv \neg\neg z \supset z$ cannot be proved in the intuitionistic propositional logic by providing a Heyting algebra which falsifies ϕ.

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Consider the Heyting algebra $0 < 1/2 < 1$, and interpret z as $1/2$. Since $\neg z$ is interpreted in 0, $\neg\neg z$ is interpreted in 1. Thus $\neg\neg z \supset z$ is true if and only if $1 \leq 1/2$, which is evidently false.

25/05/2018 (3)

Show that intuitionistic propositional logic plus $\neg\neg A = A$ yields $\neg\neg(A \vee \neg A)$.
(Hint: prove that $\neg(x \vee y) = \neg x \wedge \neg y$ holds in intuitionistic logic.)

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Following the hint:

$$
\begin{prooftree}
\AxiomC{$[x \vee y]^1$}
\AxiomC{$[x]^2$}
\AxiomC{$[\neg x \wedge \neg y]^3$}
\RightLabel{$\wedge \mathrm{E}_1$}
\UnaryInfC{$\neg x$}
\RightLabel{$\neg \mathrm{E}$}
\BinaryInfC{$\bot$}
\AxiomC{$[y]^2$}
\AxiomC{$[\neg x \wedge \neg y]^3$}
\RightLabel{$\wedge \mathrm{E}_2$}
\UnaryInfC{$\neg y$}
\RightLabel{$\neg \mathrm{E}$}
\BinaryInfC{$\bot$}
\RightLabel{$\vee \mathrm{E}^2$}
\TrinaryInfC{$\bot$}
\RightLabel{$\neg \mathrm{I}^1$}
\UnaryInfC{$\neg(x \vee y)$}
\RightLabel{$\supset \mathrm{I}^3$}
\UnaryInfC{$\neg x \wedge \neg y \supset \neg(x \vee y)$}
\end{prooftree}
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$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(x\vee y)]^1 \quad \dfrac{[x]^2}{x\vee y}\,\vee\mathrm{I}_1}{\bot}\,\neg\mathrm{E}}{\neg x}\,\neg\mathrm{I}^2 \qquad \dfrac{\dfrac{[\neg(x\vee y)]^1 \quad \dfrac{[y]^3}{x\vee y}\,\vee\mathrm{I}_2}{\bot}\,\neg\mathrm{E}}{\neg y}\,\neg\mathrm{I}^3}{\neg x\wedge\neg y}\,\wedge\mathrm{I}}{\neg(x\vee y)\supset\neg x\wedge\neg y}\,\supset\mathrm{I}^1$$

Hence $\neg\neg(A \vee \neg A) = \neg(\neg A \wedge \neg\neg A) = \neg(\neg A \wedge A) = \neg\bot = \top$.

10/01/2019 (3)

Illustrate the Gödel-Gentzen translation. Show two formulae which are equivalent in classical logic but not equivalent in intuitionistic logic.

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The first part of the exercise is Definition 19.2 in the slides. As for the second part, it suffices to take $A \vee \neg A$ and $(A \vee \neg A)^N$ in the propositional fragment: by Proposition 19.4, the latter is provable in intuitionistic logic, while the former is not even valid because of the Completeness Theorem and Fact 20.4. Hence, they are not intuitionistically equivalent. However, by Proposition 19.3, these two formulae are equivalent in classical logic.

10/01/2020 (1)

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The proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{[a \supset (b \supset c)]^1 \quad \dfrac{[a \wedge b]^2}{a}\wedge\mathrm{E}_1}{b \supset c}\supset\mathrm{E} \qquad \dfrac{[a \wedge b]^2}{b}\wedge\mathrm{E}_2}{c}\supset\mathrm{E}}{a \wedge b \supset c}\supset\mathrm{I}^2}{(a \supset (b \supset c)) \supset (a \wedge b \supset c)}\supset\mathrm{I}^1$$

The corresponding typed term via the Curry-Howard isomorphism is

$$\lambda x : (a \to (b \to c)).\lambda y : a \times b.(x(\pi_1 y))(\pi_2 y) : (a \to (b \to c)) \to (a \times b \to c)$$

Limiting Results

- 08/06/2017, third exercise
- 05/02/2019, first exercise
- 15/06/2021, third exercise

08/06/2017 (3)

Discuss whether a theory of real numbers allows to prove any true first-order sentence on $\mathbb{R}$, the set of real numbers.

08/06/2017 (3)

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Let T be a first-order theory having $\mathbb{R}$ as a model. If T is both effective and able to represent all the recursive functions then, by Gödel's incompleteness theorem, there is a formula which is both true and non provable.
Otherwise, when T is effective but does not represent all the recursive functions, there is a sentence, representing some recursive function, which cannot be derived in T.
If we drop the requirement to be effective, the theory which comprehends all the true formulae on $\mathbb{R}$, is obviously able to prove all of them, trivially.

05/02/2019 (1)

Write a proof in natural deduction using the axioms of Peano arithmetic that every $n \in \mathbb{N}$ is equal or greater than zero, where $n \geq m$ is defined as $\exists x. x + m = n$.
Discuss what happens if one defines $n \geq m$ as $\exists x. n = m + x$.

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Discuss what happens if one defines $n \geq m$ as $\exists x. n = m + x$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x. 0 + x = x}\,\text{ax}}{0 + n = n}\,\forall\text{E}}{\exists x. 0 + x = n}\,\exists\text{I}}{\forall n. \exists x. 0 + x = n}\,\forall\text{I}
$$

If one changes the definition of $\geq$ as suggested, one has, e.g., to prove that the commutative law for addition holds, which requires induction.

15/06/2021 (3)

Show that it is impossible to prove the Completeness Theorem for first-order classical logic by constructing a canonical model $\mathfrak{M}$ that is also classifying, i.e., such that, every other model can be obtained from $\mathfrak{M}$ by a function which preserves truth.
(Hint: It suffices to show that it is impossible for a specific theory.)

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(Hint: It suffices to show that it is impossible for a specific theory.)

Suppose there is such a model $\mathfrak{M}$ for Peano arithmetic. By Gödel's Incompleteness Theorem, there is a sentence G such that $\nvdash G$ and $\nvdash \neg G$. However, $\mathfrak{M}$ interprets G either as true or false.
Suppose G is true in $\mathfrak{M}$. Then it has to be true in every other model $\mathfrak{N}$ of Peano arithmetic, since $\mathfrak{M}$ is classifying and thus there is $f : \mathfrak{M} \to \mathfrak{N}$ which preserves truth, forcing G to be true also in $\mathfrak{N}$. However, by the Completeness Theorem, this fact implies $\vdash G$, getting a contradiction.
So, $\mathfrak{M}$ has to make G false, that is, $\neg G$ is true. As above, $\neg G$ has to be true in every model of Peano arithmetic, because $\mathfrak{M}$ is a classifying model. Hence, by the Completeness Theorem, $\vdash \neg G$, getting another contradiction.
Therefore, $\mathfrak{M}$ cannot exist.