MATHEMATICAL LOGIC — ASSIGNMENT ONE

(1) Prove $\vdash (\neg P \supset P) \vee (P \supset \neg P)$.

$$\dfrac{\dfrac{}{P \vee \neg P}\,\text{lem} \qquad \dfrac{\dfrac{[P]^1}{\neg P \supset P}\,{\supset}\text{I}}{(\neg P \supset P) \vee (P \supset \neg P)}\,\vee\text{I}_1 \qquad \dfrac{\dfrac{[\neg P]^1}{P \supset \neg P}\,{\supset}\text{I}}{(\neg P \supset P) \vee (P \supset \neg P)}\,\vee\text{I}_2}{(\neg P \supset P) \vee (P \supset \neg P)}\,\vee\text{E}^1$$

(2) Show that in every bounded distributive complemented lattice each element has a unique complement.

This is Proposition 5.13 in the slides.

(3) Let T be a topological space. Consider the collection

$$C = \{S \subseteq X \colon S \text{ is open in } T \text{ and } S \text{ is closed in } T\} \quad .$$

Prove that $\langle C; \subseteq \rangle$ is a Boolean algebra.

First, observe how $\langle C; \subseteq \rangle$ is evidently an order.

Fix $A, B \in C$. Observe how the least set U such that $A \subseteq U$ and $B \subseteq U$ is $A \cup B$. Symmetrically, observe how the greatest set V such that $V \subseteq A$ and $V \subseteq B$ is $A \cap B$. Note how the union and intersection of two open (closed) sets is open (closed). Hence $A \wedge B = A \cap B \in C$, $A \vee B = A \cup B \in C$, and $\langle C; \subseteq \rangle$ is a lattice.

Then, observe how $\emptyset, X \in C$, and $\emptyset \subseteq A \subseteq X$ for every $A \in C$. Hence $\bot = \emptyset$, $\top = X$, and $\langle C; \subseteq \rangle$ is a bounded lattice.

Observe how for every $A \in C$, $X \setminus A$ is open, since it is the complement of the closed set A, and $X \setminus A$ is closed, being the complement of the open set A. Thus $X \setminus A \in C$. Also, $A \cap (X \setminus A) = \emptyset$ and $A \cup (X \setminus A) = X$. Hence, $\neg A = X \setminus A$, and $\langle C; \subseteq \rangle$ is a complemented lattice.

Finally, observe how $\cap$ distributes over $\cup$ on sets in general, and thus it does so specifically on the elements of C. Hence $\langle C; \subseteq \rangle$ is a distributive lattice.

Therefore, $\langle C; \subseteq \rangle$ is a bounded distributive complemented lattice, that is, a Boolean algebra.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: Apr 4^{th}, 2022.