MATHEMATICAL LOGIC — NOTES

MARCO BENINI

1. Lecture 1: History

The history of Mathematics is the background to justify why and how Mathematical Logic took the present form.

Logic is as old as Mathematics. In fact, sound reasoning is an essential element of Mathematics.

The fundamental idea of *theorem*, a statement which holds because its truth is achieved by correct reasoning, has been introduced by the Pythagorean school.

Important to mention are: Heraclites, who introduced the term *logos* to mean reasoning; Zeno of Elea, who introduced *reduction ad absurdum*, that is, proof by contradiction, and analysed the idea of infinity in his famous paradoxes.

A turning point was Aristotle's *Organon* in which he introduced the theory of *syllogisms*, the first formal system. His work emphasises the formal nature of reasoning, and he was the first to deal with the principles of contradiction and excluded middle in a systematic way.

The stoic school, mainly Chrisyppus, introduced the notion of *modality*, the theory of conditionals, which lead to a systematic understanding of implication as a connective, and the relation between meaning and truth, posing the basis for semantics.

Centuries later, after the fall of the Roman Empire, and when Islam saw its golden age, the works of Al-Farabi, Ibn Sina (Avicenna) and others introduced further developments of the Greek tradition, in particular, identifying the basis of modal and temporal logics. In particular, Avicenna's *ma'na*, the notion of sign in the mind that does not necessarily represent an existing thing, is the first attempt to capture the nature of abstract or ideal objects.

Although interesting, we skip the Medieval logic as its influence to Mathematical Logic is minor, but it was fundamental to frame the basis of scientific reasoning.

The Greek geometric tradition and the algebraic tradition, mainly developed in the Islamic world and in the early Italian school, merged

Date: 2020–21.

together in the outstanding work of Descartes, with his analytic geometry. The importance of his contribution is difficult to overestimate: the notions of *curve* and *figure* became first class objects, which could be described by equations, no longer limiting the domain of geometry to lines, circles, and conics. Also, the notion of *space* is a natural consequence of Descartes' work: it is the set of points that can be described by *coordinates*, tuples of numbers of fixed length. It was then possible to describe multi-dimensional spaces and to smoothly apply the algebraic methods to them.

Then Newton and Liebnitz invented mathematical analysis. For Newton, this was the language in which to express his theory of gravitation, in fact, the milestone that marks the born of Science. In fact, Newton developed the *calculus of fluctions* by basing it on Euclidean geometry, trying to make it well founded, justified in the Euclid's sense. Oppositely Liebnitz wanted an agile system: today, we still use his notation for integrals and derivatives. His more intuitive approach, based on infinitesimals, was extremely influencing, and, in fact, dominated the development of Mathematics till the 19^{th} century.

Liebnitz also conceived the idea of *characteristica universalis*, an attempt to devise a formal theory of thought, allowing to mechanically calculate the act of reasoning. This idea anticipate symbolic logic with striking insights, like using the prime factorisation theorem to code propositions, anticipating Gödel's numbering.

At the beginning of the 18^{th} century, analysis was dominating Mathematics, and it has become a very abstract field, a character that became predominant in Mathematics. This race for abstraction revealed a number of deep problems: functions in their full generality were defeating intuition, casting doubts on proofs. In fact, most of the stunning achievements of analysis simply failed for general functions, showing that limits, continuity, derivation and integration are subtle concepts which require precise definitions. In turn, these definitions failed because mathematicians realised that the very notion of real number was inadequately understood.

This crisis eventually led to modern mathematical analysis: Bolzano's, Dedekind's, Cauchy's constructions of real numbers out of rationals; the notion of limit, continuity and the definition of derivatives and integrals in formal terms.

This immense effort due to Gauss, Cauchy, Weierstrass, Riemann, Dedekind and many others cleared the foundations of analysis, but also showed that many fundamental mathematical concept, took for granted, required further analysis.

In the first half of the 19^{th} century, a revolution started in the mathematical world: the advent of abstract algebra, and the discovery of non-Euclidean geometries.

In algebra, Abel and Galois developed group theory, providing a deep insight on the solutions of polynomial equations. In fact, they were the first ones to provide *limiting results*: they showed that no general algebraic method can exist for solving polynomial equations of degree greater than four.

In geometry, Gauss, Bolyai, and Lobachevsky developed alternative models of spaces in which the parallel postulate does not hold. Then, Riemann vastly generalised this idea showing how every space admits a gemotery which best describes it, whose lines are the *geodetics*, the curves of minimal length between a pair of points. Klein, later, has shown the bridge between geometry and algebra: a geometry is, in fact, a group of transformations acting on a space.

The sudden raise in abstraction, the availability of novel and powerful instruments in algebra, and the pressure from analysis led to a deeper study of the fundamental ingredients of mathematical thinking: space, numbers, and sets.

The need for a general notion of space going beyond the Euclidean intuition eventually led to topology. The study of numbers led to abstract away what was not needed in their definition, to keep the essential properties in a domain: the process led to abstract algebra, the notions of groups, rings, fields, and vector spaces. The study of sets, by Cantor is particularly relevant to us. By comparing sets through functions, Cantor discovered that the idea of infinite is not unique: he showed that real numbers are more numerous than natural numbers, leading to the notion of *cardinality*.

In algebra, we have to mention the work of Boole, who introduced an algebraic system to represent logical propositions in the sense of Aristotle. This is the starting point, together with Cantor's work, of a new discipline: Mathematical Logic.

Gottlob Frege in his Begriffschrift (1879) introduced variables, quantifiers, and a rigorous treatment of functions basing on intuitive set theory.

His purpose was to show that arithmetic is a branch of logic, and no intuition is needed to understand it, an approach which takes the name of Logicism in philosophy.

Also, his work can be considered the first attempt to provide a rigorous foundation to the whole Mathematics.

A principle used y Frege is the axiom of unlimited comprehension: if P is a formula depending on just x, then $\{x\colon P(x)\}$ is a set. In 1903, Bertrand Russell wrote a letter to Free in which his famous paradox shows how Frege's system is inconsistent, as it contains a contradiction. The contradiction cannot be avoided but dropping that axiom.

In 1889, Giuseppe Peano defined a formal theory for arithmetic which bears his name. He and Dedekind recognised induction as the characterising principle of natural numbers.

In 1899 David Hilbert developed a complete set of axioms for Euclidean geometry (Grundlagen der Geometrie), which makes formal Euclid's *Elements*.

In 1900, at the International Conference of Mathematicians, David Hilbert posed a list of 23 problems. The first two required to resolve the Continuum Hypothesis, and to prove the consistency of elementary arithmetic. The tenth problem asked for a method to decide whether a multivariate polynomial equation over integers has a solution.

The solutions to these problems and their consequences are some of the fundamental results in Mathematical Logic.

Ernst Zermelo in 1908 introduced a formal system for set theory, in which the Axiom of Choice is stated for the first time. Later, Abraham Frænkel added the Axiom of Replacement obtaining ZFC, which is usually regarderd as the reference for formal set theory.

In 1910 , the first volume of Principia Mathematicae by Russell and Whiteead appeared. This monumental work is an attempt to reconstruct the whole Mathematics from a simple formal system, avoiding internal contradictions. It is based on a peculiar form of type theory.

Leopold Löwenheim (1915) and Thoralf Skolem (1920) obtained limiting results, summarised in their Theorem, saying that first order theories cannot influence the cardinality of their infinite models. Skolem went as far to realise that a formal set theory must have a countable model, which is completely counterintuitive.

In 1929, Kurt Gödel proved the Completeness Theorem in his doctoral dissertation. As a consequence, he derived compactness, which proves the finitary nature of first order systems.

In 1931, *On formally undecidable propositions of Principia Mathematicae and Related Systems* was published. This is a milestone in human knowledge. In that work, Gödel proved that every sufficiently strong yet effective system is either inconsistent of contains a true but unprovable statement. He also showed that the consistency of such a formal system is one of these unprovable statements, in fact, closing the efforts of Russell and Whitehead's quest for a universal system.

In 1936, Gerard Gentzen proved the consistency of Peano arithmetic in a finitary system together with the principle of transfinite induction up to the ε_0 ordinal, which can be considered a measure of the proving power of arithmetic. This result introduced also cut-elimination. In fact, this is the starting point of Proof Theory.

In the '20s, the notion of computability was in the air. Early works, like the definition of functions by Schönfinkel, eventually led Gödel, Turing, Kleene and Church to recognise that effective systems and computable functions are two faces of the same coin. In particular, the notion of computable function proved to be very robust, leading to the Church-Turing thesis, and eventually generated an entirely new discipline: Computer Science.

It is worth noting how the solution of the Halting Problem, posed by Hilbert in 1928, is strictly related to Gödel's incompleteness result and the existence of non-computable functions.

An important turning point in Logic was Intuitionism, a philosophical line of thought promoted by Brouwer and formalised by Heyting. This line emphasises *constructions* as the building blocks of Mathematics. In time, it turned out that intuitionistic logic is strictly related with computability and plays a fundamental role in the foundation of Mathematics, for example, in topos theory.

At his point in history, around 1940, it becomes difficult to keep track of developments and achievements: Mathematical Logic became a mature, deeply and widely studied field with many branches and ramifications.

We limit ourselves to mention the impact of category theory, which ultimately leads to abandon sets in favour of a more abstract but also more regular algebraic structure; the impact of type theories as the computational counterparts of constructive formal systems which have recently been shown to have strict links with algebraic topology in Homotopy Type Theory; the development of ordinal analysis to classify theories and problems according to their proving power, essentially extending and generalising the limiting results; the immense amount of logics which have been studied, modal, temporal, epistemic, deontic, paraconsistent, many-valued, fuzzy, just to name a few with their semantics, proof theorym and an ever growing amount of applications in every field.