MATHEMATICAL LOGIC — ASSIGNMENT FOUR

(1) Prove in Peano Arithmetic that $\forall x. (x = 0) \vee (\exists y. x = \mathsf{S}\, y)$.

(Hint: a sketch of a proof in natural deduction is enough.)

Call $P = (x = 0) \vee (\exists y. x = \mathsf{S}\, y)$ and $I = P[0/x] \wedge (\forall x. P \supset P[\mathsf{S}\, x/x])$.

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\dfrac{\dfrac{}{I \supset \forall x. P}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{0 = 0}\,\text{ax}}{P[0/x]}\,{\vee}\text{I}_1 \qquad \dfrac{\dfrac{\dfrac{[P]^1 \qquad \begin{matrix}[x = 0]^2 \\ \vdots\ \text{a} \\ P[\mathsf{S}\, x/x]\end{matrix} \qquad \begin{matrix}[\exists y. x = \mathsf{S}\, y]^2 \\ \vdots\ \text{b} \\ P[\mathsf{S}\, x/x]\end{matrix}}{P[\mathsf{S}\, x/x]}\,{\vee}\text{E}^2}{P \supset P[\mathsf{S}\, x/x]}\,{\supset}\text{I}^1}{\forall x. P \supset P[\mathsf{S}\, x/x]}\,\forall\text{I}}{I}\,{\wedge}\text{I}}{\forall x. P}\,{\supset}\text{E}
$$

The a subproof is easily solved posing $y = 0$; the b subproof follows proving $\mathsf{S}\, x = \mathsf{S}\,\mathsf{S}\, y$ which is a consequence of its hypothesis.

(2) Write all the axioms of Peano Arithmetic.

See slides 428–430.

(3) Show that it is impossible to prove the Completeness Theorem for first-order classical logic by constructing a canonical model $\mathfrak{M}$ that is also classifying, i.e., such that, every other model can be obtained from $\mathfrak{M}$ by a function which preserves truth.

(Hint: It suffices to show that it is impossible for a specific theory.)

Suppose there is such a model $\mathfrak{M}$ for Peano arithmetic. By Gödel's Incompleteness Theorem, there is a sentence G such that $\nvdash G$ and $\nvdash \neg G$.

However, $\mathfrak{M}$ interprets G either as true or false.

Suppose G is true in $\mathfrak{M}$. Then it has to be true in every other model $\mathfrak{N}$ of Peano arithmetic, since $\mathfrak{M}$ is classifying and thus there is $f\colon \mathfrak{M} \to \mathfrak{N}$ which preserves truth, forcing G to be true also in $\mathfrak{N}$. However, by the Completeness Theorem, this fact implies $\vdash G$, getting a contradiction.

So, $\mathfrak{M}$ has to make G false, that is, $\neg G$ is true. As above, $\neg G$ has to be true in every model of Peano arithmetic, because $\mathfrak{M}$ is a classifying model. Hence, by the Completeness Theorem, $\vdash \neg G$, getting another contradiction.

Therefore, $\mathfrak{M}$ cannot exist.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: January 28^{th}, 2020.