MATHEMATICAL LOGIC — ASSIGNMENT THREE

(1) Prove in intuitionistic propositional logic that $(a \supset (b \supset c)) \supset (a \wedge b \supset c)$, and translate this proof into a term in the simple theory of types.

The proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{[a \supset (b \supset c)]^1 \qquad \dfrac{[a \wedge b]^2}{a}\,{\scriptstyle \wedge\mathrm{E}_1}}{b \supset c}\,{\scriptstyle \supset\mathrm{E}} \qquad \dfrac{[a \wedge b]^2}{b}\,{\scriptstyle \wedge\mathrm{E}_2}}{c}\,{\scriptstyle \supset\mathrm{E}}}{a \wedge b \supset c}\,{\scriptstyle \supset\mathrm{I}^2}}{(a \supset (b \supset c)) \supset (a \wedge b \supset c)}\,{\scriptstyle \supset\mathrm{I}^1}$$

The corresponding typed term via the Curry-Howard isomorphism is

$$\lambda x : (a \to (b \to c)).\, \lambda y : a \times b.\, (x\,(\pi_1 y))(\pi_2 y) : (a \to (b \to c)) \to (a \times b \to c)$$

(2) State and prove the Schröder-Bernstein Theorem.

See Theorem 11.1 in the slides.

(3) Show that the Axiom of Choice implies the Law of Excluded Middle.

[Hint: Let P be a proposition and let x be a variable not appearing in P. Define $U = \{x \in \{0,1\} \colon P \vee (x = 0)\}$ and $V = \{x \in \{0,1\} \colon P \vee (x = 1)\}$. There must be a choice function on $\{U, V\}$, hence...]

...there is f such$f(U) \in U \wedge f(V) \in V$. By the definition of U and V, it holds that $(P \vee f(U) = 0) \wedge (P \vee f(V) = 1)$, which is equivalent to $P \vee f(U) \neq f(V)$ (*).

This is equivalent to $\neg P \supset f(U) \neq f(V)$ (**).

Notice that $P \supset U = V$, so $P \supset f(U) = f(V)$, or equivalently, $f(U) \neq f(V) \supset \neg P$.

Combining with (**), $\neg P$ is equivalent to $f(U) \neq f(V)$.

Hence, rewriting (*), $P \vee \neg P$ as required.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: January 10^{th}, 2020.