MATHEMATICAL LOGIC — ASSIGNMENT TWO

(1) Prove that $A \vee \exists x.\, B = \exists x.\, (A \vee B)$ when $x \notin \mathsf{FV}(A)$.

$$\dfrac{\dfrac{[A \vee \exists x.\, B]^1 \quad \dfrac{\dfrac{[A]^2}{A \vee B}\,\vee\mathrm{I}_1}{\exists x.\, A \vee B}\,\exists\mathrm{I} \quad \dfrac{[\exists x.\, B]^2 \quad \dfrac{\dfrac{[B]^3}{A \vee B}\,\vee\mathrm{I}_2}{\exists x.\, A \vee B}\,\exists\mathrm{I}}{\exists x.\, A \vee B}\,\exists\mathrm{E}^3}{\exists x.\, A \vee B}\,\vee\mathrm{E}^2}{(A \vee \exists x.\, B) \supset \exists x.\, (A \vee B)}\,\supset\mathrm{I}^1$$

$$\dfrac{\dfrac{[\exists x.\, A \vee B]^1 \quad \dfrac{[A \vee B]^2 \quad \dfrac{[A]^3}{A \vee \exists x.\, B}\,\vee\mathrm{I}_1 \quad \dfrac{\dfrac{[B]^3}{\exists x.\, B}\,\exists\mathrm{I}}{A \vee \exists x.\, B}\,\vee\mathrm{I}_2}{A \vee \exists x.\, B}\,\vee\mathrm{E}^3}{A \vee \exists x.\, B}\,\exists\mathrm{E}^2}{(\exists x.\, A \vee B) \supset A \vee \exists x.\, B}\,\supset\mathrm{I}^1$$

(2) State the Löwenheim-Skolem Theorem.

This is Theorem 10.6 in the slides.

(3) Define an alternative model of integers in which there is an infinite number.

Take the language of the theory $\mathbb{Z}$ of integer numbers and extend it with the constant ∞. Consider the following axioms:

$$\phi_k \equiv \infty > k$$

for every $k \in \mathbb{N}$. Call $\Phi = \{\phi_k : k \in \mathbb{N}\}$.

Let Ξ be any finite subset of $\mathbb{Z} \cup \Phi$.

If $\Xi \cap \Phi = \emptyset$, then Ξ has the integer numbers as a model in which ∞ is interpreted in, let say, 42.

Otherwise, $\Xi \cap \Phi \neq \emptyset$ and it is finite. Hence there is m such that $\phi_m \in \Xi$ and $\phi_i \notin \Xi$ for every $i > m$. Then Ξ has the integer numbers as a model, posing $\infty = m + 1$.

Therefore, $\mathbb{Z} \cup \Phi$ has a model $\mathcal{M}$ by the Compactness Theorem. In particular, $\mathcal{M}$ is a model for $\mathbb{Z}$, as required.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: November 27^{th}, 2019.