MATHEMATICAL LOGIC — ASSIGNMENT FOUR

(1) Write a proof in natural deduction using the axioms of Peano arithmetic that every $n \in \mathbb{N}$ is equal or greater than zero, where $n \geq m$ is defined as $\exists x.\, x + m = n$.

Discuss what happens if one defines $n \geq m$ as $\exists x.\, n = m + x$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\, 0 + x = x}\,\text{ax}}{0 + n = n}\,\forall\text{E}}{\exists x.\, 0 + x = n}\,\exists\text{I}}{\forall n.\, \exists x.\, 0 + x = n}\,\forall\text{I}$$

If one changes the definition of $\geq$ as suggested, one has, e.g., to prove that the commutative law for addition holds, which requires induction.

(2) Show that there is a non-standard model for Peano arithmetic.

This is Proposition 19.3 in the slides.

(3) Consider the Completeness Theorem for classical first order logic. Show that there is no proof of it which constructs a canonical model $\mathfrak{M}$ for a theory T that is also classifying, i.e., such that, every other model of T can be obtained from $\mathfrak{M}$ by a function which preserves truth.

Suppose there is such a model $\mathfrak{M}$ for Peano arithmetic. By Gödel's Incompleteness Theorem, there is a sentence G such that $\nvdash G$ and $\nvdash \neg G$.

However, $\mathfrak{M}$ interprets G either as true or false. Suppose G is true in $\mathfrak{M}$. Then it has to be true in every other model $\mathfrak{N}$ of Peano arithmetic, since $\mathfrak{M}$ is a classifying model and thus there is $f\colon \mathfrak{M} \to \mathfrak{N}$ which preserves truth, forcing G to be true also in $\mathfrak{N}$. However, by the Completeness Theorem, this fact implies $\vdash G$, getting a contradiction.

So, $\mathfrak{M}$ makes G false, that is, $\neg G$ is true. As above, $\neg G$ has to be true in every model of Peano arithmetic, because $\mathfrak{M}$ is a classifying model. Hence, by the Completeness Theorem, $\vdash \neg G$, getting another contradiction.

Therefore, $\mathfrak{M}$ cannot exist.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

Date: February 5$^{\text{th}}$, 2019.