MATHEMATICAL LOGIC — ASSIGNMENT TWO

(1) Prove that $(A \lor \forall x.\, B) \supset \forall x.\, (A \lor B)$ when $x \notin \mathsf{FV}(A)$. Find a counterexample if $x \in \mathsf{FV}(A)$.

$$\dfrac{\dfrac{[A \lor \forall x.\, B]^1 \qquad \dfrac{\dfrac{[A]^2}{A \lor B}\,\lor\mathrm{I}_1}{\forall x.\, A \lor B}\,\forall\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{[\forall x.\, B]^2}{B}\,\forall\mathrm{E}}{A \lor B}\,\lor\mathrm{I}_2}{\forall x.\, A \lor B}\,\forall\mathrm{I}}{\forall x.\, A \lor B}\,\lor\mathrm{E}^2}{(A \lor \forall x.\, B) \supset \forall x.\, A \lor B}\,\supset\mathrm{I}^1$$

It is worth noticing that the first occurrence of the $\forall$ introduction rule depends on the hypothesis.

Consider the property $P(x)$ which stands for x *is prime* in the theory of natural numbers. Then the following is an instance of the statement:

$$P(x) \lor \forall x.\, P(x) \supset \forall x.\, P(x) \lor P(x) \ .$$

The consequent is equivalent to $\forall x.\, P(x)$, which is clearly false, since, e.g., 4 is not prime.

However, $P(3)$ holds since 3 is prime, thus $P(3) \lor \forall x.\, P(x)$ is true. Hence, the implication does not hold for every instance.

(2) State and prove the Compactness Theorem.

This is Theorem 10.1 in the slides.

(3) Define an alternative model of real numbers in which there is constant ∞ such that $1/\infty = 0$.

Take the language of the theory $\mathcal{R}$ of real numbers and extend it with the constant ∞. Consider the following axioms:

$$\phi_k \equiv -\frac{1}{k+1} < \frac{1}{\infty} \land \frac{1}{\infty} < \frac{1}{k+1}$$

for every $k \in \mathbb{N}$. Call $\Phi = \{\phi_k : k \in \mathbb{N}\}$.

Let Ξ be any finite subset of $\mathcal{R} \cup \Phi$. If $\Xi \cap \Phi = \emptyset$, Ξ has the real numbers as model in which ∞ is interpreted in, let say, 42. Otherwise, $\Xi \cap \Phi \neq \emptyset$ and it is finite. Hence there is m such that $\phi_m \in \Xi$ and $\phi_i \notin \Xi$ for every $i > m$. Then Ξ has the real numbers as model, posing $\infty = m + 2$. Therefore, $\mathcal{R} \cup \Phi$ has a model $\mathcal{M}$ by the Compactness Theorem.

In $\mathcal{M}$, $1/\infty < 1/(k+1)$ for every $k \in \mathbb{N}$, so $1/\infty \leq 0$. Also $1/\infty > -1/(k+1)$ for every $k \in \mathbb{N}$, so $1/\infty \geq 0$. Hence, by anti-symmetry, $1/\infty = 0$.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.

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