Mathematical Logic — Assignment One

March 23$^{\text{th}}$, 2018

1. Prove that $\vdash \neg\neg(A \supset B) = \neg\neg A \supset \neg\neg B$

$$
\begin{prooftree}
\AxiomC{$[A]^1$}
\AxiomC{$[A \supset B]^2$}
\RightLabel{$\supset$E}
\BinaryInfC{B}
\AxiomC{$[\neg B]^3$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^2}
\UnaryInfC{$\neg(A \supset B)$}
\AxiomC{$[\neg\neg(A \supset B)]^4$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^1}
\UnaryInfC{$\neg A$}
\AxiomC{$[\neg\neg A]^5$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^3}
\UnaryInfC{$\neg\neg B$}
\RightLabel{$\supset$I^5}
\UnaryInfC{$\neg\neg A \supset \neg\neg B$}
\RightLabel{$\supset$I^4}
\UnaryInfC{$\neg\neg(A \supset B) \supset \neg\neg A \supset \neg\neg B$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[\neg(A \supset B)]^1$}
\AxiomC{$[\neg\neg A \supset \neg\neg B]^2$}
\AxiomC{$[A]^3$}
\AxiomC{$[\neg A]^4$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\supset$I^4}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\supset$E}
\BinaryInfC{$\neg\neg B$}
\AxiomC{$[\neg(A \supset B)]^1$}
\AxiomC{$[B]^5$}
\RightLabel{$\supset$I}
\UnaryInfC{$A \supset B$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^5}
\UnaryInfC{$\neg B$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\bot$E}
\UnaryInfC{B}
\RightLabel{$\supset$I^3}
\UnaryInfC{$A \supset B$}
\RightLabel{$\neg$E}
\BinaryInfC{$\bot$}
\RightLabel{$\neg$I^1}
\UnaryInfC{$\neg\neg(A \supset B)$}
\RightLabel{$\supset$I^2}
\UnaryInfC{$(\neg\neg A \supset \neg\neg B) \supset \neg\neg(A \supset B)$}
\end{prooftree}
$$

2. Show that, in any bounded, distributive and complemented lattice, each element x has a unique complement.

   This is Proposition 5.13, slide 102.

   Suppose the element x has two complements y and z. Then, by definition of complement

- $x \wedge y = \bot = x \wedge z$,
- $x \vee y = \top = x \vee z$.

Thus, $y = y \wedge \top = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = \bot \vee (y \wedge z) = (z \wedge x) \vee (z \wedge y) = z \wedge (x \vee y) = z \wedge \top = z$.

3. The Law of Excluded Middle has been presented by means of the following inference rule:

$$\frac{}{A \vee \neg A}\,\text{lem}$$

for any formula A.

Consider instead the following inference rule:

$$\frac{\begin{matrix}[\neg A]^1 \\ \vdots \\ \bot\end{matrix}}{A}\,\text{alt}^1$$

Is the alt inference rule sound? Discuss the question and motivate your answer.

Suppose to have a proof $\neg A \vdash \bot$: then the alt rule can be simulated by

$$\frac{\dfrac{}{A \vee \neg A}\,\text{lem} \qquad [A]^1 \qquad \dfrac{\begin{matrix}[\neg A]^1 \\ \vdots \\ \bot\end{matrix}}{A}\,\bot\text{E}}{A}\,\vee\text{E}^1$$

Hence it is sound.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.