Mathematical Logic — Assignment Two

December 5th, 2016

1. Prove that $\vdash \neg\neg(\forall x.\, A) \supset (\forall x.\, \neg\neg A)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x.\, A]^1}{A}\,\forall\mathrm{E} \qquad [\neg A]^2}{\bot}\,\neg\mathrm{E}}{\neg\forall x.\, A}\,\neg\mathrm{I}^1 \qquad [\neg\neg\forall x.\, A]^3}{\bot}\,\neg\mathrm{E}}{\neg\neg A}\,\neg\mathrm{I}^2}{\forall x.\, \neg\neg A}\,\forall\mathrm{I}}{\neg\neg(\forall x.\, A) \supset \forall x.\, \neg\neg A}\,\to\mathrm{I}^3$$

2. Show that, for any set X, there is a bijective correspondence between X and $|X|$, using the Axiom of Choice.

   This is Theorem 13.30 (Well ordering) in slide 309.

3. Consider the signature with addition, multiplication, 0, 1, and the $<$ relation, and let $\mathbb{R}$ be the real numbers with the standard interpretation of the symbols. Let T be the set of all true sentences (formulas with no free variables) in the structure $\mathbb{R}$ on that signature. Show that T has a model in which there are infinitesimal numbers, i.e. there is c for which $0 < c < 1/n$ for every $n \in \mathbb{N}$.

   Define $\phi_n \equiv 0 < c \wedge c < 1/n$ for some new constant c and $n \in \mathbb{N}$, $n > 1$. Call $T_I = T \cup \{\phi_n \colon n > 1\}$.

   Consider any finite subset T_0 of T_I. It is immediate to see that $\mathbb{R}$ is a model for T_0 since it is a model for T and, calling $N = \max\{n \in \mathbb{N} \colon \phi_n \in T_0\}$, clearly $c = 1/(N+1)$ satisfies all the $\phi_n \in T_0$ (notice how N is undefined when T_0 does not contain any ϕ_n, but it does not matter). So, by the Compactness Theorem, T_I has a model in which there is an infinitesimal c (in fact, there are infinite, since each $c + x$ is infinitesimal too).

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.