Mathematical Logic — Assignment One

November 7^{th}, 2016

1. Prove that $\vdash (A \supset B) = (\neg B \supset \neg A)$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[A \supset B]^1 \quad [A]^2}{B}{\scriptstyle \to\mathrm{E}} \qquad [\neg B]^3}{\bot}{\scriptstyle \neg\mathrm{E}}}{\neg A}{\scriptstyle \neg\mathrm{I}^2}}{\neg B \supset \neg A}{\scriptstyle \to\mathrm{I}^3}}{(A \supset B) \supset (\neg B \supset \neg A)}{\scriptstyle \to\mathrm{I}^1}
$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{B \vee \neg B}{\scriptstyle \mathrm{lem}} \qquad [B]^1 \qquad \cfrac{\cfrac{\cfrac{[\neg B \supset \neg A]^2 \quad [\neg B]^1}{\neg A}{\scriptstyle \to\mathrm{E}} \qquad [A]^3}{\bot}{\scriptstyle \neg\mathrm{E}}}{B}{\scriptstyle \bot\mathrm{E}}}{B}{\scriptstyle \vee\mathrm{E}^1}}{A \supset B}{\scriptstyle \to\mathrm{I}^3}}{(\neg B \supset \neg A) \supset (A \supset B)}{\scriptstyle \to\mathrm{I}^2}
$$

2. Show that, in any bounded, distributive and complemented lattice, each element x has a unique complement.

   This is Proposition 5.13, slide 102.

   Suppose the element x has two complements y and z. Then, by definition of complement

   - $x \wedge y = \bot = x \wedge z$,
   - $x \vee y = \top = x \vee z$.

   Thus, $y = y \wedge \top = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = \bot \vee (y \wedge z) = (z \wedge x) \vee (z \wedge y) = z \wedge (x \vee y) = z \wedge \top = z$.

3. The Law of Excluded Middle has been presented by means of the following inference rule:

$$\frac{}{A \vee \neg A}\,\text{lem}$$

for any formula A.

Consider instead the following inference rule:

$$\frac{}{x \vee \neg x}\,\text{lemr}$$

for any **variable** x.

Does the lemr inference rule allow to derive all the theorems of propositional logic?

Discuss the question and motivate your answer.

By induction on the structure of A, we can easily show that $A \vee \neg A$ is derivable using just the lemr inference rule.

So, by substituting each occurrence of thelem inference rule with the proof as above, we can transform any proof $\pi : \Gamma \vdash B$ into a proof $\pi^r : \Gamma \vdash B$ that uses just the lemr inference rule.

Since the lemr inference rule is an instance of the lem rule, the two deductive systems allow to prove exactly the same theorems.

Each question is worth 12 points. The points in all the four assignments will be added together and the result will be divided by 4, and this will be the final result. Remember to mark your answer sheet with your name.