

MATHEMATICAL LOGIC 2015/16

1. EXERCISES

- (1) $\vdash A \supset (B \supset A)$
- (2) $\vdash (A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
- (3) $\vdash A = \neg\neg A$
- (4) $\vdash \neg A = \neg\neg\neg A$
- (5) $\vdash \neg(A \vee B) = \neg A \wedge \neg B$
- (6) $\vdash (A \supset B) \vee (B \supset A)$
- (7) $\vdash \neg A \supset (A \supset B)$
- (8) $\vdash ((A \supset B) \supset A) \supset A$
- (9) $\vdash \neg\neg(A \supset B) = \neg\neg A \supset \neg\neg B$
- (10) $\vdash A \supset B = \neg B \supset \neg A$
- (11) $\vdash A \vee B = \neg A \supset B$
- (12) $\vdash A \supset B = \neg(A \wedge \neg B)$
- (13) $\vdash A \wedge B \supset \neg(A \supset \neg B)$
- (14) $\vdash \forall x. B \supset A = B \supset \forall x. A$ with $x \notin \text{FV}(B)$
- (15) $\vdash \forall x. A \supset B = (\exists x. A) \supset B$ with $x \notin \text{FV}(B)$
- (16) $\vdash \neg\neg(A \wedge B) = \neg\neg A \wedge \neg\neg B$
- (17) $\vdash \neg\neg\forall x. A \supset \forall x. \neg\neg A$
- (18) $\vdash A \wedge \exists x. B = \exists x. A \wedge B$ with $x \notin \text{FV}(A)$
- (19) $\vdash A \wedge \forall x. B = \forall x. A \wedge B$ with $x \notin \text{FV}(A)$
- (20) $\vdash A \vee \exists x. B = \exists x. A \vee B$ with $x \notin \text{FV}(A)$
- (21) $\vdash (\forall x. P \vee \neg P) \wedge \neg\forall x. \neg P \supset \exists x. P$
- (22) $\vdash \forall x. A \wedge B = (\forall x. A) \wedge (\forall x. B)$
- (23) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \wedge (\exists x. B)$
- (24) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B)$
- (25) $\vdash \exists x. A \vee B = (\exists x. A) \vee (\exists x. B)$
- (26) $\vdash (A \vee B) \vee C = A \vee (B \vee C)$
- (27) $\vdash (A \wedge B) \wedge C = A \wedge (B \wedge C)$
- (28) $\vdash A \vee B = B \vee A$
- (29) $\vdash A \wedge B = B \wedge A$
- (30) $\vdash A \vee A = A$
- (31) $\vdash A \wedge A = A$
- (32) $\vdash A \vee (A \wedge B) = A$
- (33) $\vdash A \wedge (A \vee B) = A$
- (34) $\vdash A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
- (35) $\vdash A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
- (36) $\vdash \neg(A \vee B) = \neg A \wedge \neg B$
- (37) $\vdash \neg(A \wedge B) = \neg A \vee \neg B$
- (38) $\vdash A \wedge B = \neg(\neg A \vee \neg B)$
- (39) $\vdash A \vee B = \neg(\neg A \wedge \neg B)$
- (40) $\vdash A \supset B = \neg A \vee B$

$$(41) \vdash (\forall x. A) \supset \exists x. A$$

$$(42) \vdash (\exists x. \forall y. A) \supset \forall y. \exists x. A$$

$$(43) \vdash (\exists x. A) = \neg \forall x. \neg A$$

$$(44) \vdash (\forall x. A) = \neg \exists x. \neg A$$

2. SOLUTIONS

(1) $\vdash A \supset (B \supset A)$

$$\frac{\frac{[A]^1}{B \supset A} \supset I}{A \supset (B \supset A)} \supset I^1$$

(2) $\vdash (A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

$$\frac{\frac{[A \vee B]^1 \quad \frac{[A]^2 \quad [A \supset C]^3}{C} \supset E \quad \frac{[B]^2 \quad [B \supset C]^4}{C} \supset E}{C} \vee E^2}{\frac{C}{A \vee B \supset C} \supset I^1} \supset I^4}{(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))} \supset I^3$$

(3) $\vdash A = \neg\neg A$

$$\frac{\frac{A \vee \neg A}{A} \text{lem} \quad \frac{[A]^1}{A} \supset I^2}{\neg\neg A \supset A} \supset I^2 \quad \frac{\frac{[\neg A]^1 \quad [\neg\neg A]^2}{\perp} \neg E \quad \frac{\perp}{A} \perp E}{\neg\neg A} \vee E^1 \quad \frac{\frac{[\neg A]^1 \quad [A]^2}{\perp} \neg E}{\neg\neg A} \supset I^1}{A \supset \neg\neg A} \supset I^2$$

(4) $\vdash \neg A = \neg\neg\neg A$

$$\frac{\frac{[\neg\neg\neg A]^1 \quad \frac{[\neg A]^2 \quad [A]^3}{\perp} \neg E}{\neg\neg A} \supset I^2 \quad \frac{\perp}{\neg A} \perp E}{\neg\neg\neg A} \supset I^1 \quad \frac{[\neg\neg A]^1 \quad [\neg A]^2}{\perp} \neg E}{\neg A \supset \neg\neg\neg A} \supset I^2$$

(5) $\vdash \neg(A \vee B) = \neg A \wedge \neg B$

$$\frac{\frac{[\neg(A \vee B)]^1 \quad \frac{[A]^2}{A \vee B} \vee I_1}{\perp} \neg E \quad \frac{\perp}{\neg A} \neg I^2}{\neg A} \neg I^2 \quad \frac{[\neg(A \vee B)]^1 \quad \frac{[B]^3}{A \vee B} \vee I_2}{\perp} \neg E \quad \frac{\perp}{\neg B} \neg I^3}{\neg B} \neg I^3}{\neg A \wedge \neg B} \wedge I}{\neg(A \vee B) \supset \neg A \wedge \neg B} \supset I^1$$

$$\frac{[A \vee B]^1 \frac{[A]^2 \frac{[\neg A \wedge \neg B]^3}{\neg A} \wedge E_1}{\perp} \neg E \quad \frac{[B]^2 \frac{[\neg A \wedge \neg B]^3}{\neg B} \wedge E_2}{\perp} \neg E}{\perp} \vee E^2}{\frac{\perp}{\neg(A \vee B)} \neg I^1} \neg I^3$$

(6) $\vdash (A \supset B) \vee (B \supset A)$

$$\frac{A \vee \neg A \text{ lem} \quad \frac{[A]^1}{B \supset A} \supset I \quad \frac{[A]^2 \quad [\neg A]^1}{\perp} \neg E}{(A \supset B) \vee (B \supset A)} \vee I_2 \quad \frac{\frac{\perp}{B} \perp E}{A \supset B} \supset I^2}{(A \supset B) \vee (B \supset A)} \vee I_1 \quad \vee E^1$$

(7) $\vdash \neg A \supset (A \supset B)$

$$\frac{\frac{[\neg A]^1 \quad [A]^2}{\perp} \neg E}{B} \perp E}{A \supset B} \supset I^2}{\neg A \supset (A \supset B)} \supset I^1$$

(8) $\vdash ((A \supset B) \supset A) \supset A$

$$\frac{A \vee \neg A \text{ lem} \quad [A]^1 \quad \frac{[(A \supset B) \supset A]^2}{A} \supset E \quad \frac{[\neg A]^1 \quad [A]^3}{\perp} \neg E}{A} \vee E^1}{((A \supset B) \supset A) \supset A} \supset I^2$$

(9) $\vdash \neg\neg(A \supset B) = \neg\neg A \supset \neg\neg B$

$$\frac{[A]^1 \quad [A \supset B]^2}{B} \supset E \quad \frac{[\neg B]^3}{\perp} \neg E}{\neg(A \supset B)} \neg I^2 \quad \frac{\perp}{[\neg\neg(A \supset B)]^4} \neg E}{\neg A} \neg I^1 \quad \frac{[\neg\neg A]^5}{\perp} \neg E}{\neg\neg B} \neg I^3}{\neg\neg A \supset \neg\neg B} \supset I^5}{\neg\neg(A \supset B) \supset \neg\neg A \supset \neg\neg B} \supset I^4$$

$$\frac{\frac{\frac{[A]^3 \quad [\neg A]^4}{\perp} \neg E \quad \frac{[\neg(A \supset B)]^1 \quad \frac{[B]^5}{A \supset B} \supset I}{\neg E}}{\neg \neg A} \supset I^4 \quad \frac{\perp}{\neg B} \neg I^5}{\neg \neg B} \supset E}{\frac{\perp}{B} \perp E} \supset I^3 \quad \frac{[\neg(A \supset B)]^1}{A \supset B} \supset I^3}{\perp} \neg I^1 \quad \frac{(\neg \neg A \supset \neg \neg B) \supset \neg \neg(A \supset B)}{\neg \neg(A \supset B)} \supset I^2$$

(10) $\vdash A \supset B = \neg B \supset \neg A$

$$\frac{\frac{\frac{[A \supset B]^1 \quad [A]^2}{B} \supset E \quad \frac{[\neg B]^3}{\perp} \neg E}{\neg A} \neg I^2 \quad \frac{\neg B \supset \neg A}{(A \supset B) \supset (\neg B \supset \neg A)} \supset I^1}{\frac{[\neg B \supset \neg A]^2 \quad [\neg B]^1}{\neg A} \supset E \quad \frac{[A]^3}{\perp} \neg E}{B \vee \neg B} \text{lem} \quad \frac{[B]^1}{B} \perp E}{A \supset B} \supset I^3 \quad \frac{(\neg B \supset \neg A) \supset (A \supset B)}{(A \supset B) \supset (\neg B \supset \neg A)} \supset I^2$$

(11) $\vdash A \vee B = \neg A \supset B$

$$\frac{\frac{\frac{[A \vee B]^1 \quad \frac{[A]^2 \quad [\neg A]^3}{\perp} \neg E}{B} \perp E}{\neg A \supset B} \supset I^3 \quad \frac{A \vee B}{A \vee B \supset (\neg A \supset B)} \supset I^1}{\frac{[A]^1 \quad \frac{[\neg A]^1 \quad [\neg A \supset B]^2}{B} \supset E}{A \vee B} \vee I_1 \quad \frac{A \vee B}{A \vee B} \vee I_2}{A \vee \neg A} \text{lem} \quad \frac{A \vee B}{(\neg A \supset B) \supset A \vee B} \supset I^2$$

(12) $\vdash A \supset B = \neg(A \wedge \neg B)$

$$\frac{\frac{\frac{[A \supset B]^1 \quad \frac{[A \wedge \neg B]^2}{A} \wedge E_1}{B} \supset E \quad \frac{[A \wedge \neg B]^2}{\neg B} \wedge E_2}{\perp} \neg E}{\neg(A \wedge \neg B)} \neg I^2 \quad \frac{(\neg(A \wedge \neg B) \supset \neg(A \wedge \neg B))}{(A \supset B) \supset \neg(A \wedge \neg B)} \supset I^1}{\frac{[A]^2 \quad [\neg B]^1}{A \wedge \neg B} \wedge I \quad \frac{[\neg(A \wedge \neg B)]^3}{\perp} \neg E}{B \vee \neg B} \text{lem} \quad \frac{[B]^1}{B} \perp E}{A \supset B} \supset I^2 \quad \frac{(\neg(A \wedge \neg B) \supset (A \supset B))}{\neg(A \wedge \neg B) \supset (A \supset B)} \supset I^3$$

(13) $\vdash A \wedge B \supset \neg(A \supset \neg B)$

$$\frac{\frac{\frac{[A \supset \neg B]^1}{\neg B} \quad \frac{[A \wedge B]^2}{A} \wedge E_1}{\supset E} \quad \frac{[A \wedge B]^2}{B} \wedge E_2}{\neg E} \quad \frac{\perp}{\neg(A \supset \neg B)} \neg I^1}{A \wedge B \supset \neg(A \supset \neg B)} \supset I^2$$

(14) $\vdash \forall x. B \supset A = B \supset \forall x. A$ with $x \notin \text{FV}(B)$

$$\frac{\frac{[B]^1 \quad \frac{[\forall x. B \supset A]^2}{B \supset A} \forall E}{\supset E} \quad \frac{A}{\forall x. A} \forall I}{B \supset \forall x. A} \supset I^1}{(\forall x. B \supset A) \supset (B \supset \forall x. A)} \supset I^2 \quad \frac{\frac{[B \supset \forall x. A]^1 \quad [B]^2}{\forall x. A} \supset E}{\supset E} \quad \frac{A}{B \supset A} \supset I^2}{\forall x. B \supset A} \forall I}{(B \supset \forall x. A) \supset (\forall x. B \supset A)} \supset I^1$$

(15) $\vdash \forall x. A \supset B = (\exists x. A) \supset B$ with $x \notin \text{FV}(B)$

$$\frac{\frac{[\exists x. A]^1 \quad \frac{[A]^2 \quad \frac{[\forall x. A \supset B]^3}{A \supset B} \forall E}{\supset E} \quad B}{\exists E^2} \quad \frac{B}{(\exists x. A) \supset B} \supset I^1}{(\forall x. A \supset B) \supset ((\exists x. A) \supset B)} \supset I^3 \quad \frac{\frac{[A]^1}{\exists x. A} \exists I \quad [(\exists x. A) \supset B]^2}{\supset E} \quad \frac{B}{A \supset B} \supset I^1}{\forall x. A \supset B} \forall I}{((\exists x. A) \supset B) \supset (\forall x. A \supset B)} \supset I^2$$

(16) $\vdash \neg\neg(A \wedge B) = \neg\neg A \wedge \neg\neg B$

$$\frac{\frac{[\neg\neg(A \wedge B)]^1 \quad \frac{[A \wedge B]^3}{A} \wedge E_1}{\supset E} \quad \frac{\perp}{\neg(A \wedge B)} \neg I^3}{\neg E} \quad \frac{\perp}{\neg\neg A} \neg I^2 \quad \frac{[\neg\neg(A \wedge B)]^1 \quad \frac{[A \wedge B]^5}{B} \wedge E_2}{\supset E} \quad \frac{\perp}{\neg(A \wedge B)} \neg I^5}{\neg E} \quad \frac{\perp}{\neg\neg B} \neg I^4}{\neg I^4} \quad \frac{\neg\neg A \wedge \neg\neg B}{\neg\neg(A \wedge B) \supset \neg\neg A \wedge \neg\neg B} \supset I^1$$

$$\begin{array}{c}
 \frac{[A]^1 [B]^2}{A \wedge B} \wedge I \quad \frac{[\neg(A \wedge B)]^3}{\perp} \neg E \\
 \frac{\perp}{\neg A} \neg I^1 \quad \frac{[\neg\neg A \wedge \neg\neg B]^4}{\neg\neg A} \wedge E_1 \\
 \frac{\perp}{\neg B} \neg I^2 \quad \frac{[\neg\neg A \wedge \neg\neg B]^4}{\neg\neg B} \wedge E_2 \\
 \frac{\perp}{\neg\neg(A \wedge B)} \neg I^3 \\
 \frac{\neg\neg A \wedge \neg\neg B \supset \neg\neg(A \wedge B)}{\neg\neg A \wedge \neg\neg B \supset \neg\neg(A \wedge B)} \supset I^4
 \end{array}$$

(17) $\vdash \neg\neg\forall x. A \supset \forall x. \neg\neg A$

$$\begin{array}{c}
 \frac{[\forall x. A]^1}{A} \forall E \quad \frac{[\neg A]^2}{\perp} \neg E \\
 \frac{\perp}{\neg\forall x. A} \neg I^1 \quad \frac{[\neg\neg\forall x. A]^3}{\perp} \neg E \\
 \frac{\perp}{\neg\neg A} \neg I^2 \quad \frac{\forall x. \neg\neg A}{\neg\neg(\forall x. A)} \forall I \\
 \frac{\neg\neg(\forall x. A) \supset \forall x. \neg\neg A}{\neg\neg(\forall x. A) \supset \forall x. \neg\neg A} \supset I^3
 \end{array}$$

(18) $\vdash A \wedge \exists x. B = \exists x. A \wedge B$ with $x \notin \text{FV}(A)$

$$\begin{array}{c}
 \frac{[A \wedge \exists x. B]^1}{A} \wedge E_1 \quad \frac{[B]^2}{A \wedge B} \wedge I \\
 \frac{[A \wedge \exists x. B]^1}{\exists x. B} \wedge E_2 \quad \frac{A \wedge B}{\exists x. A \wedge B} \exists I \\
 \frac{\exists x. A \wedge B}{A \wedge (\exists x. B) \supset \exists x. A \wedge B} \supset I^1 \\
 \frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[A \wedge B]^2}{B} \wedge E_2 \\
 \frac{[A \wedge B]^2}{\exists x. B} \exists I \quad \frac{A \wedge \exists x. B}{A \wedge \exists x. B} \wedge I \\
 \frac{[\exists x. A \wedge B]^1}{A \wedge \exists x. B} \exists E^2 \\
 \frac{(\exists x. A \wedge B) \supset A \wedge \exists x. B}{(\exists x. A \wedge B) \supset A \wedge \exists x. B} \supset I^1
 \end{array}$$

(19) $\vdash A \wedge \forall x. B = \forall x. A \wedge B$ with $x \notin \text{FV}(A)$

$$\begin{array}{c}
 \frac{[A \wedge \forall x. B]^1}{A} \wedge E_1 \quad \frac{[A \wedge \forall x. B]^1}{\forall x. B} \wedge E_2 \\
 \frac{A \wedge B}{\forall x. A \wedge B} \forall I \\
 \frac{\forall x. A \wedge B}{A \wedge (\forall x. B) \supset \forall x. A \wedge B} \supset I^1 \\
 \frac{[\forall x. A \wedge B]^1}{A \wedge B} \forall E \\
 \frac{[\forall x. A \wedge B]^1}{A} \wedge E_1 \quad \frac{[\forall x. A \wedge B]^1}{B} \wedge E_2 \\
 \frac{A \wedge \forall x. B}{\forall x. B} \forall I \\
 \frac{A \wedge \forall x. B}{(\forall x. A \wedge B) \supset A \wedge \forall x. B} \supset I^1
 \end{array}$$

(20) $\vdash A \vee \exists x. B = \exists x. A \vee B$ with $x \notin \text{FV}(A)$

$$\frac{[\exists x. A \vee B]^1 \quad \frac{[A \vee B]^2 \quad \frac{[A]^3}{A \vee \exists x. B} \vee I_1 \quad \frac{[B]^3}{\exists x. B} \exists E}{A \vee \exists x. B} \vee I_2}{A \vee \exists x. B} \vee E^3}{\frac{A \vee \exists x. B}{(\exists x. A \vee B) \supset A \vee \exists x. B} \supset I^1} \exists E^2$$

$$\frac{[A \vee \exists x. B]^1 \quad \frac{[A]^2}{A \vee B} \vee I_1 \quad \frac{[\exists x. B]^2}{\exists x. A \vee B} \exists I}{\exists x. A \vee B} \exists I^1 \quad \frac{[B]^3}{A \vee B} \vee I_2 \quad \frac{[\exists x. A \vee B]}{\exists x. A \vee B} \exists E^3}{\exists x. A \vee B} \vee E^2}{\frac{A \vee (\exists x. B) \supset \exists x. A \vee B} \supset I^1} \supset I^1$$

(21) $\vdash (\forall x. P \vee \neg P) \wedge \neg \forall x. \neg P \supset \exists x. P$

$$\frac{(\exists x. P) \vee \neg \exists x. P \text{ lem} \quad [\exists x. P]^1 \quad \frac{\perp}{\exists x. P} \perp E}{\exists x. P} \vee E^1 \quad \frac{[\forall x. P \vee \neg P] \wedge \neg \forall x. \neg P}{\neg \forall x. \neg P} \wedge E_1 \quad \frac{[\neg \exists x. P]^1}{\neg \exists x. P} \neg E \quad \frac{\perp}{\neg P} \neg I^3 \quad \frac{\perp}{\forall x. P} \forall I}{\forall x. P} \forall I}{\frac{(\forall x. P \vee \neg P) \wedge \neg \forall x. \neg P \supset \exists x. P} \supset I^2} \supset I^2$$

(22) $\vdash \forall x. A \wedge B = (\forall x. A) \wedge (\forall x. B)$

$$\frac{\frac{[\forall x. A \wedge B]^1}{A \wedge B} \forall E \quad \frac{[\forall x. A \wedge B]^1}{A \wedge B} \forall E}{\frac{A \wedge B}{A} \wedge E_1 \quad \frac{A \wedge B}{B} \wedge E_2}{\frac{\forall x. A}{\forall x. A} \forall I \quad \frac{\forall x. B}{\forall x. B} \forall I}{(\forall x. A) \wedge (\forall x. B)} \wedge I}{(\forall x. A \wedge B) \supset (\forall x. A) \wedge (\forall x. B)} \supset I^1$$

$$\frac{[(\forall x. A) \wedge (\forall x. B)]^1}{\forall x. A} \wedge E_1 \quad \frac{[(\forall x. A) \wedge (\forall x. B)]^1}{\forall x. B} \wedge E_2}{\frac{\forall x. A}{A} \forall E \quad \frac{\forall x. B}{B} \forall E}{A \wedge B} \wedge I}{\frac{A \wedge B}{\forall x. A \wedge B} \forall I}{(\forall x. A) \wedge (\forall x. B) \supset \forall x. A \wedge B} \supset I^1$$

(23) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \wedge (\exists x. B)$

$$\frac{\frac{\frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[A \wedge B]^2}{B} \wedge E_2}{\exists x. A} \exists I \quad \frac{\frac{[A \wedge B]^2}{B} \wedge E_2}{\exists x. B} \exists I}{\frac{[\exists x. A \wedge B]^1}{(\exists x. A) \wedge (\exists x. B)} \wedge I} \exists E^2 \quad \frac{(\exists x. A) \wedge (\exists x. B)}{(\exists x. A \wedge B) \supset (\exists x. A) \wedge (\exists x. B)} \supset I^1$$

(24) $\vdash (\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B)$ Both these proofs are correct:

$$\frac{\frac{\frac{[A \wedge B]^2}{A} \wedge E_1}{\exists x. A} \exists I \quad \frac{[\exists x. A \wedge B]^1}{(\exists x. A) \vee (\exists x. B)} \vee I_1}{\frac{(\exists x. A) \vee (\exists x. B)}{(\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B)} \supset I^1} \exists E^2 \quad \frac{\frac{\frac{[A \wedge B]^2}{B} \wedge E_2}{\exists x. B} \exists I \quad \frac{[\exists x. A \wedge B]^1}{(\exists x. A) \vee (\exists x. B)} \vee I_2}{\frac{(\exists x. A) \vee (\exists x. B)}{(\exists x. A \wedge B) \supset (\exists x. A) \vee (\exists x. B)} \supset I^1} \exists E^2$$

(25) $\vdash \exists x. A \vee B = (\exists x. A) \vee (\exists x. B)$

$$\frac{\frac{[\exists x. A \vee B]^1}{(\exists x. A) \vee (\exists x. B)} \vee E^3 \quad \frac{\frac{[A]^3}{\exists x. A} \exists I \quad \frac{[B]^3}{\exists x. B} \exists I}{(\exists x. A) \vee (\exists x. B)} \vee I_1}{\frac{(\exists x. A) \vee (\exists x. B)}{(\exists x. A \vee B) \supset (\exists x. A) \vee (\exists x. B)} \supset I^1} \exists E^2$$

$$\frac{\frac{[(\exists x. A) \vee (\exists x. B)]^1}{\exists x. A \vee B} \exists E^3 \quad \frac{\frac{[A]^3}{A \vee B} \vee I_1 \quad \frac{[B]^4}{A \vee B} \vee I_2}{\exists x. A \vee B} \exists I}{\frac{(\exists x. A) \vee (\exists x. B)}{(\exists x. A) \vee (\exists x. B) \supset (\exists x. A \vee B)} \supset I^1} \exists E^2$$

(26) $\vdash (A \vee B) \vee C = A \vee (B \vee C)$

$$\frac{\frac{[(A \vee B) \vee C]^1}{A \vee (B \vee C)} \vee E^3 \quad \frac{\frac{[A]^3}{A \vee (B \vee C)} \vee I_1 \quad \frac{[B]^3}{B \vee C} \vee I_1}{A \vee (B \vee C)} \vee E^3 \quad \frac{\frac{[C]^2}{B \vee C} \vee I_2}{A \vee (B \vee C)} \vee E^3}{\frac{A \vee (B \vee C)}{(A \vee B) \vee C \supset A \vee (B \vee C)} \supset I^1} \vee E^2$$

$$\frac{[A \vee (B \vee C)]^1 \frac{\frac{[A]^2}{A \vee B} \vee I_1 \quad \frac{[B \vee C]^2 \frac{\frac{[B]^3}{A \vee B} \vee I_2 \quad \frac{[C]^3}{(A \vee B) \vee C} \vee I_2}{(A \vee B) \vee C} \vee I_1}{(A \vee B) \vee C} \vee E^2}}{(A \vee B) \vee C} \vee E^2}}{A \vee (B \vee C) \supset (A \vee B) \vee C} \supset I^1$$

(27) $\vdash (A \wedge B) \wedge C = A \wedge (B \wedge C)$

$$\frac{\frac{\frac{[(A \wedge B) \wedge C]^1}{A \wedge B} \wedge E_1 \quad \frac{[(A \wedge B) \wedge C]^1}{A \wedge B} \wedge E_1 \quad \frac{[(A \wedge B) \wedge C]^1}{C} \wedge E_2}{\frac{A}{A} \wedge E_1 \quad \frac{B \wedge C}{B \wedge C} \wedge I}}{A \wedge (B \wedge C)} \wedge I}{(A \wedge B) \wedge C \supset A \wedge (B \wedge C)} \supset I^1$$

$$\frac{\frac{\frac{[A \wedge (B \wedge C)]^1}{A} \wedge E_1 \quad \frac{[A \wedge (B \wedge C)]^1}{B \wedge C} \wedge E_2 \quad \frac{[A \wedge (B \wedge C)]^1}{B \wedge C} \wedge E_2}{\frac{A \wedge B}{A \wedge B} \wedge I \quad \frac{C}{C} \wedge E_2}}{(A \wedge B) \wedge C} \wedge I}{A \wedge (B \wedge C) \supset (A \wedge B) \wedge C} \supset I^1$$

(28) $\vdash A \vee B = B \vee A$ We notice that the property is auto-dual

$$\frac{[A \vee B]^1 \frac{[A]^2}{B \vee A} \vee I_2 \quad \frac{[B]^2}{B \vee A} \vee I_1}{B \vee A} \vee E^2}}{A \vee B \supset B \vee A} \supset I^1$$

(29) $\vdash A \wedge B = B \wedge A$ We notice that the property is auto-dual

$$\frac{\frac{[A \wedge B]^1}{B} \wedge E_2 \quad \frac{[A \wedge B]^1}{A} \wedge E_1}{B \wedge A} \wedge I}}{A \wedge B \supset B \wedge A} \supset I^1$$

(30) $\vdash A \vee A = A$

$$\frac{[A \vee A]^1 \quad [A]^2 \quad [A]^2}{A} \vee E^2}}{A \vee A \supset A} \supset I^1 \quad \frac{[A]^1}{A \vee A} \vee I_1 \quad \frac{[A]^1}{A \vee A} \vee I_2}{A \supset A \vee A} \supset I^1$$

(31) $\vdash A \wedge A = A$

$$\frac{\frac{[A \wedge A]^1}{A} \wedge E_1}{A \wedge A \supset A} \supset I^1 \quad \frac{\frac{[A \wedge A]^1}{A} \wedge E_2}{A \wedge A \supset A} \supset I^1 \quad \frac{\frac{[A]^1 [A]^1}{A \wedge A} \wedge I}{A \supset A \wedge A} \supset I^1$$

(32) $\vdash A \vee (A \wedge B) = A$

$$\frac{\frac{[A \vee (A \wedge B)]^1 [A]^2}{A} \vee E^2 \quad \frac{[A \wedge B]^2}{A} \wedge E_1}{A \vee (A \wedge B) \supset A} \supset I^1 \quad \frac{\frac{[A]^1}{A \vee (A \wedge B)} \vee I_1}{A \supset A \vee (A \wedge B)} \supset I^1$$

(33) $\vdash A \wedge (A \vee B) = A$

$$\frac{\frac{[A \wedge (A \vee B)]^1}{A} \wedge E_1}{A \wedge (A \vee B) \supset A} \supset I^1 \quad \frac{\frac{[A]^1}{A \vee B} \vee I_1}{A \wedge (A \vee B)} \wedge I}{A \supset A \wedge (A \vee B)} \supset I^1$$

(34) $\vdash A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

$$\frac{\frac{[A \wedge (B \vee C)]^1}{B \vee C} \wedge E_2 \quad \frac{\frac{[A \wedge (B \vee C)]^1}{A} \wedge E_1 \quad [B]^2}{A \wedge B} \wedge I \quad \frac{[A \wedge (B \vee C)]^1}{A} \wedge E_1 \quad [C]^2}{A \wedge C} \wedge I}{\frac{(A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \supset (A \wedge B) \vee (A \wedge C)} \supset I^1} \supset I^1$$

$$\frac{[(A \wedge B) \vee (A \wedge C)]^1 \quad \frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[A \wedge B]^2}{B \vee C} \vee I_1 \quad \frac{[A \wedge C]^2}{A} \wedge E_1 \quad \frac{[A \wedge C]^2}{B \vee C} \vee I_2}{\frac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C) \supset A \wedge (B \vee C)} \supset I^1} \supset I^1$$

(35) $\vdash A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

$$\frac{[A \vee (B \wedge C)]^1 \quad \frac{[A]^2}{A \vee B} \vee I_1 \quad \frac{[A]^2}{A \vee C} \vee I_1 \quad \frac{[B \wedge C]^2}{B} \wedge E_1 \quad \frac{[B \wedge C]^2}{C} \wedge E_2}{\frac{(A \vee B) \wedge (A \vee C)}{A \vee (B \wedge C) \supset (A \vee B) \wedge (A \vee C)} \supset I^1} \supset I^1$$

$$\frac{\frac{[(A \vee B) \wedge (A \vee C)]^1}{A \vee B} \wedge E_1 \quad \frac{[A]^2 \quad \frac{[(A \vee B) \wedge (A \vee C)]^1}{A \vee C} \wedge E_2 \quad \frac{[A]^3}{A \vee (B \wedge C)} \vee I_1 \quad \frac{[B]^2 [C]^3}{B \wedge C} \wedge I}{A \vee (B \wedge C)} \vee I_2}{\frac{A \vee (B \wedge C)}{A \vee (B \wedge C)} \vee E_2} \vee E_3}{\frac{A \vee (B \wedge C)}{(A \vee B) \wedge (A \vee C) \supset A \vee (B \wedge C)} \supset I^1} \supset I^1$$

$$(36) \vdash \neg(A \vee B) = \neg A \wedge \neg B$$

$$\frac{\frac{[A]^1}{A \vee B} \vee I_1 \quad \frac{[\neg(A \vee B)]^2}{\perp} \neg E}{\frac{\perp}{\neg A \wedge \neg B} \perp E} \neg E \quad \frac{\frac{[B]^3}{A \vee B} \vee I_2 \quad \frac{[\neg(A \vee B)]^2}{\perp} \neg E}{\frac{\perp}{\neg A \wedge \neg B} \perp E} \neg E \quad \frac{[\neg A]^1 \quad [\neg B]^3}{\neg A \wedge \neg B} \wedge I}{\frac{\neg A \wedge \neg B}{\neg A \wedge \neg B} \vee E^1} \vee E^3}{\frac{\neg A \wedge \neg B}{\neg(A \vee B) \supset \neg A \wedge \neg B} \supset I^2} \supset I^2$$

$$\frac{[A \vee B]^1 \quad \frac{[A]^2 \quad \frac{[\neg A \wedge \neg B]^3}{\neg A} \wedge E_1}{\perp} \neg E \quad \frac{[B]^2 \quad \frac{[\neg A \wedge \neg B]^3}{\neg B} \wedge E_2}{\perp} \neg E}{\frac{\perp}{\neg(A \vee B)} \neg I^1} \vee E^2}{\frac{\neg(A \vee B)}{\neg A \wedge \neg B \supset \neg(A \vee B)} \supset I^3} \supset I^3$$

$$(37) \vdash \neg(A \wedge B) = \neg A \vee \neg B$$

$$\frac{\frac{[A]^1 [B]^2}{A \wedge B} \wedge I \quad \frac{[\neg(A \wedge B)]^3}{\perp} \neg E}{\frac{\perp}{\neg A \vee \neg B} \perp E} \neg E \quad \frac{[\neg B]^2}{\neg A \vee \neg B} \vee I_2 \quad \frac{[\neg A]^1}{\neg A \vee \neg B} \vee I_1}{\frac{\neg A \vee \neg B}{\neg A \vee \neg B} \vee E^2} \vee E^2}{\frac{\neg A \vee \neg B}{\neg(A \wedge B) \supset \neg A \vee \neg B} \supset I^3} \supset I^3$$

$$\frac{[\neg A \vee \neg B]^1 \quad \frac{[A \wedge B]^2}{A} \wedge E_1 \quad \frac{[\neg A]^3}{\perp} \neg E \quad \frac{[A \wedge B]^2}{B} \wedge E_2 \quad \frac{[\neg B]^3}{\perp} \neg E}{\frac{\perp}{\neg(A \wedge B)} \neg I^2} \vee E^3}{\frac{\neg(A \wedge B)}{\neg A \vee \neg B \supset \neg(A \wedge B)} \supset I^1} \supset I^1$$

(38) $\vdash A \wedge B = \neg(\neg A \vee \neg B)$

$$\begin{array}{c}
 \frac{[\neg A \vee \neg B]^1 \quad \frac{[\neg A]^2 \quad \frac{[A \wedge B]^3}{A} \wedge E_1}{\perp} \neg E \quad \frac{[\neg B]^2 \quad \frac{[A \wedge B]^3}{B} \wedge E_2}{\perp} \neg E}{\perp} \vee E^2}{\frac{\perp}{\neg(\neg A \vee \neg B)} \neg I^1} \supset I^3 \\
 \frac{A \wedge B \supset \neg(\neg A \vee \neg B)}{\neg(\neg A \vee \neg B) \supset A \wedge B} \supset I^3 \\
 \frac{A \vee \neg A \text{ lem} \quad \frac{B \vee \neg B \text{ lem} \quad \frac{[A]^1 \quad [B]^2}{A \wedge B} \wedge I \quad \frac{\perp}{A \wedge B} \perp E}{\frac{A \wedge B}{\neg(\neg A \vee \neg B) \supset A \wedge B} \supset I^3} \vee E^2 \quad \frac{\frac{[\neg A]^1}{\neg A \vee \neg B} \vee I_1 \quad \frac{[\neg(\neg A \vee \neg B)]^3}{\neg E} \quad \frac{\perp}{A \wedge B} \perp E}{\frac{A \wedge B}{\neg(\neg A \vee \neg B) \supset A \wedge B} \supset I^3} \vee E^1} \vee E^1}
 \end{array}$$

 (39) $\vdash A \vee B = \neg(\neg A \wedge \neg B)$

$$\begin{array}{c}
 \frac{[A \vee B]^1 \quad \frac{[\neg A \wedge \neg B]^2}{\neg A} \wedge E_1 \quad [A]^3 \quad \frac{[\neg A \wedge \neg B]^2}{\neg B} \wedge E_2 \quad [B]^3}{\perp} \neg E}{\perp} \vee E^3}{\frac{\perp}{\neg(\neg A \wedge \neg B)} \neg I^2} \supset I^1 \\
 \frac{A \vee B \supset \neg(\neg A \wedge \neg B)}{\neg(\neg A \wedge \neg B) \supset A \vee B} \supset I^1 \\
 \frac{A \vee \neg A \text{ lem} \quad \frac{[A]^1}{A \vee B} \vee I_1 \quad \frac{B \vee \neg B \text{ lem} \quad \frac{[B]^2}{A \vee B} \vee I_2 \quad \frac{\perp}{A \vee B} \perp E}{\frac{A \vee B}{\neg(\neg A \wedge \neg B) \supset A \vee B} \supset I^1} \vee E^2 \quad \frac{\frac{[\neg A]^1 \quad [\neg B]^2}{\neg A \wedge \neg B} \wedge I \quad [\neg(\neg A \wedge \neg B)]^3}{\neg E} \quad \frac{\perp}{A \vee B} \perp E}{\frac{A \vee B}{\neg(\neg A \wedge \neg B) \supset A \vee B} \supset I^1} \vee E^1} \vee E^1}
 \end{array}$$

 (40) $\vdash A \supset B = \neg A \vee B$

$$\begin{array}{c}
 \frac{A \vee \neg A \text{ lem} \quad \frac{[A \supset B]^1 \quad [A]^2}{B} \supset E \quad \frac{[\neg A]^2}{\neg A \vee B} \vee I_1}{\frac{\neg A \vee B}{(A \supset B) \supset \neg A \vee B} \supset I^1} \vee E^2 \quad \frac{[\neg A]^2 \quad [A]^3}{\perp} \neg E \quad \frac{[\neg A \vee B]^1 \quad \frac{\perp}{B} \perp E \quad [B]^2}{B} \vee E^2}{\frac{A \supset B}{\neg A \vee B \supset (A \supset B)} \supset I^1} \supset I^3} \supset I^1}
 \end{array}$$

 (41) $\vdash (\forall x. A) \supset \exists x. A$

$$\frac{\frac{[\forall x. A]^1}{A} \forall E \quad \frac{A}{\exists x. A} \exists I}{(\forall x. A) \supset \exists x. A} \supset I^1$$

(42) $\vdash (\exists x. \forall y. A) \supset \forall y. \exists x. A$

$$\frac{\frac{\frac{[\forall y. A]^2}{\forall E} A}{\exists x. A} \exists I}{[\exists x. \forall y. A]^1 \forall y. \exists x. A} \forall I}{\forall y. \exists x. A} \exists E^2}{(\exists x. \forall y. A) \supset \forall y. \exists x. A} \supset I^1$$

(43) $\vdash (\exists x. A) = \neg \forall x. \neg A$

$$\frac{\frac{[\exists x. A]^1 \frac{[A]^2 \frac{[\forall x. \neg A]^3}{\neg A} \forall E}{\perp} \exists E^2}{\perp} \neg I^3}{\neg \forall x. \neg A} \neg I^3}{(\exists x. A) \supset \neg \forall x. \neg A} \supset I^1}{\frac{[\neg \exists x. A]^1 \frac{[A]^2}{\exists x. A} \exists I}{\perp} \neg E}{\neg A} \neg I^2}{\forall x. \neg A} \forall I}{\forall x. \neg A} \neg I^2}{\frac{(\exists x. A) \vee \neg \exists x. A}{\exists x. A} \text{lem}} \frac{[\exists x. A]^1 \frac{[\neg(\forall x. \neg A)]^3}{\exists x. A} \neg E}{\exists x. A} \exists I}{\exists x. A} \exists I^1}{\neg(\forall x. \neg A) \supset \exists x. A} \supset I^3}$$

(44) $\vdash (\forall x. A) = \neg \exists x. \neg A$

$$\frac{\frac{[\exists x. \neg A]^1 \frac{[\neg A]^2 \frac{[\forall x. A]^3}{A} \forall E}{\perp} \exists E^2}{\perp} \neg I^1}{\neg \exists x. \neg A} \neg I^1}{(\forall x. A) \supset \neg \exists x. \neg A} \supset I^3}{\frac{A \vee \neg A}{[A]^1 \frac{[\neg(\exists x. \neg A)]^2}{\exists x. \neg A} \exists I}{\perp} \neg E} \text{lem}} \frac{[A]^1 \frac{[\neg A]}{A} \neg E}{\forall x. A} \forall I}{\forall x. A} \forall I^1}{\neg(\exists x. \neg A) \supset \forall x. A} \supset I^2}$$