Intuitionistic First-Order Logic

Categorical semantics via the Curry-Howard isomorphism

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Introduction

An observation: in part D of P. Johnstone's *Sketches of an Elephant*, there is a categorical semantics for the simply typed λ-calculus. In the very same class of models, one can give a semantics to the corresponding fragment of propositional logic.

The problem:
is it possible to do the same for full first-order intuitionistic logic?

Introduction

Johstone's account comes from Lambek and Scott's *Introduction to Higher-Order Categorical Logic*. The considered propositional logic is minimal logic limited to conjunction and implication.

In Taylor, *Practical Foundation of Mathematics*, one finds that the treatment of disjunction requires distributive categories in order to follow the same pattern as the previous works.

To my knowledge, no categorical semantics appears in literature which models the full first-order intuitionistic logic AND the corresponding λ-calculus.

The λ-calculus

Definition 1 (Lambda signature)

A λ*-signature* $\Sigma = \langle S, F, R, \mathrm{Ax} \rangle$ is a structure where

1. $\langle S, F, R \rangle$ is a logical signature, i.e.,
   1.1 a set S of *sort symbols*;
   1.2 a set F of *function symbols*, each one decorated as $f \colon s_1 \times \cdots \times s_n \to s_0$, with $s_0, \ldots, s_n \in S$;
   1.3 a set R of *relation symbols*, each one decorated as $r \colon s_1 \times \cdots \times s_n$, with $s_1, \ldots, s_n \in S$;
2. Ax is the set of *axiom symbols*, each one decorated as $a \colon A \to B$ where $A, B \in \lambda\mathrm{Types}(\Sigma)$ and $\mathrm{FV}(A \to B) = \emptyset$.

We call $\mathrm{LTerms}(\Sigma)$ the collection of logical terms constructed from the signature Σ, assuming to have a denumerable set of variables V_s for each $s \in S$.

The λ-calculus

Definition 2 (Lambda type)

Fixed a λ-signature Σ, the *λ-types on* Σ are inductively defined along with their *free variables* as follows:

1. $0, 1 \in \lambda\text{Types}(\Sigma)$ and $\text{FV}(0) = \text{FV}(1) = \emptyset$;
2. if $p : s_1 \times \cdots \times s_n \in R$ and $t_1 : s_1, \ldots, t_n : s_n \in \text{LTerms}(\Sigma)$, then $p(t_1, \ldots, t_n) \in \lambda\text{Types}(\Sigma)$ and $\text{FV}(p(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \text{FV}(t_i : s_i)$;
3. if $A, B \in \lambda\text{Types}(\Sigma)$ then $A \times B, A + B, A \to B \in \lambda\text{Types}(\Sigma)$ and $\text{FV}(A \times B) = \text{FV}(A + B) = \text{FV}(A \to B) = \text{FV}(A) \cup \text{FV}(B)$;
4. if $x \in V_s$ and $A \in \lambda\text{Types}(\Sigma)$ then $\forall x : s. A, \exists x : s. A \in \lambda\text{Types}(\Sigma)$ and $\text{FV}(\forall x : s. A) = \text{FV}(\exists x : s. A) = \text{FV}(A) \setminus \{x : s\}$.

The λ-calculus

Definition 3 (Lambda term)

Fixed a λ-signature $\Sigma = \langle S, F, R, \mathrm{Ax} \rangle$, for each type $t \in \lambda\mathrm{Types}(\Sigma)$, we assume there is a denumerable set W_t of *(typed) variables*.

A λ*-term* is inductively defined together with its free variables as:

1. if $x \in W_t$ then $x : t \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(x : t) = \{x : t\}$;
2. if $f : A \to B \in \mathrm{Ax}$ and $t : A \in \lambda\mathrm{Terms}(\Sigma)$ then $f(t) : B \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(f(t) : B) = \mathrm{FV}(t : A)$;
3. if $s : A, t : B \in \lambda\mathrm{Terms}(\Sigma)$ then $\langle s, t \rangle : A \times B \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\langle s, t \rangle : A \times B) = \mathrm{FV}(s : A) \cup \mathrm{FV}(t : B)$;
4. if $t : A \times B \in \lambda\mathrm{Types}(\Sigma)$ then $\mathrm{fst}(t) : A \in \lambda\mathrm{Terms}(\Sigma)$, $\mathrm{snd}(t) : B \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{fst}(t) : A) = \mathrm{FV}(\mathrm{snd}(t) : B) = \mathrm{FV}(t : A \times B)$;

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The λ-calculus

$\hookrightarrow$ (Lambda term)

5. if $t : A \in \lambda\mathrm{Terms}(\Sigma)$ then $\mathrm{inl}_B(t) : A + B \in \lambda\mathrm{Terms}(\Sigma)$, $\mathrm{inr}_B(t) : B + A \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{inl}_B(t) : A + B) = \mathrm{FV}(\mathrm{inr}_B(t) : B + A) = \mathrm{FV}(t : A)$;
6. if $s : A + B, t : A \to C, r : B \to C \in \lambda\mathrm{Terms}(\Sigma)$ then $\mathrm{when}(s, t, r) : C \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{when}(s, t, r) : C) = \mathrm{FV}(s : A + B) \cup \mathrm{FV}(t : A \to C) \cup \mathrm{FV}(r : B \to C)$;
7. if $x \in W_A$ and $t : B \in \lambda\mathrm{Terms}(\Sigma)$ then $(\lambda x : A.\, t) : A \to B \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}((\lambda x : A.\, t) : A \to B) = \mathrm{FV}(t : B) \setminus \{x : A\}$;
8. if $s : A \to B, t : A \in \lambda\mathrm{Terms}(\Sigma)$ then $s \cdot t : B \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(s \cdot t : B) = \mathrm{FV}(s : A \to B) \cup \mathrm{FV}(t : A)$;
9. $* : 1 \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(* : 1) = \emptyset$;
10. $\mathrm{F}_A : 0 \to A \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{F}_A : 0 \to A) = \emptyset$;

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The λ-calculus

$\hookrightarrow$ (Lambda term)

11. if $x \in V_s$ and $t : A \in \lambda\mathrm{Terms}(\Sigma)$ where $x : s \notin \mathrm{FV}^*(t : A)$, then $\mathrm{allI}(\lambda x : s.\, t) : (\forall x : s.\, A) \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{allI}(\lambda x : s.\, t) : (\forall x : s.\, A)) = \mathrm{FV}(t : A)$;
12. if $t : (\forall x : s.\, A) \in \lambda\mathrm{Terms}(\Sigma)$ and $r : s \in \mathrm{LTerms}(\Sigma)$ then $\mathrm{allE}(t, r) : (A[r/x]) \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{allE}(t, r) : (A[r/x])) = \mathrm{FV}(t : (\forall x : s.\, A))$;
13. if $x \in V_s$, $r : s \in \mathrm{LTerms}(\Sigma)$ and $t : (A[r/x]) \in \lambda\mathrm{Terms}(\Sigma)$ then $\mathrm{exI}_x(t) : (\exists x : s.\, A) \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{exI}_x(t) : (\exists x : s.\, A)) = \mathrm{FV}(t : (A[r/x]))$;
14. if $t : (\exists x : s.\, A), \in \lambda\mathrm{Terms}(\Sigma)$ and $r : A \to B \in \lambda\mathrm{Terms}(\Sigma)$ where $x : s \notin \mathrm{FV}^*(r : A \to B)$, then $\mathrm{exE}(t, (\lambda x : s.\, r)) : B \in \lambda\mathrm{Terms}(\Sigma)$ and $\mathrm{FV}(\mathrm{exE}(t, (\lambda x : s.\, r)) : B) = \mathrm{FV}(t : (\exists x : s.\, A)) \cup \mathrm{FV}(r : A \to B)$.

In the previous definition, $x : s \in \mathrm{FV}^*(t : A)$ if and only if there is $r \in \lambda\mathrm{Types}(\Sigma)$ and $y \in W_r$ such that $x : s \in \mathrm{FV}(r)$ and $y : r \in \mathrm{FV}(t : A)$.

The λ-calculus

Definition 4 (Lambda calculus)

A *derivation* is inductively defined by the following inference rules, whose antecedents and consequents are equalities-in-context within a fixed λ-signature Σ:

(eq$_0$) $\vec{x} : \vec{A}.\, s =_C t \vdash \vec{y} : \vec{B}.\, s[r_1/x_1, \ldots, r_n/x_n] =_C t[r_1/x_1, \ldots, r_n/x_n]$ where, for any $1 \le i \le n$, $\vec{y} : \vec{B}.\, r_i : A_i$ is a term-in-context;

(eq$_1$) $$\left.\begin{matrix} (\vec{x} : \vec{A}.\, s_1 =_{B_1} t_1) \\ \vdots \\ (\vec{x} : \vec{A}.\, s_m =_{B_m} t_m) \end{matrix}\right\} \vdash \vec{x} : \vec{A}.\, r[\vec{s}/\vec{y}] =_C r[\vec{t}/\vec{y}];$$

(eq$_2$) $\vdash x : A.\, x =_A x$;

(eq$_3$) $x : A, y : A.\, x =_A y \vdash x : A, y : A.\, y =_A x$;

(eq$_4$) $$\left.\begin{matrix} (x : A, y : A, z : A.\, x =_A y) \\ (x : A, y : A, z : A.\, y =_A z) \end{matrix}\right\} \vdash x : A, y : A, z : A.\, x =_A z;$$

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The λ-calculus

$\hookrightarrow$ (Lambda calculus)

(eq$_5$) $\vec{x}:\vec{A}.\, s =_C t \vdash \vec{x}:\vec{A}.\, (\lambda y:B.\, s) =_{B\to C} (\lambda y:B.\, t)$;

(eq$_6$) $\vec{x}:\vec{A}.\, r =_C t \vdash \vec{x}:\vec{A}.\, \mathrm{allI}(\lambda y:s.\, r) =_{(\forall y:s.\, C)} \mathrm{allI}(\lambda y:s.\, t)$;

(eq$_7$) $\vec{x}:\vec{A}.\, u =_C v \vdash \vec{x}:\vec{A}.\, \mathrm{exE}(t, (\lambda y:s.\, u)) =_C \mathrm{exE}(t, (\lambda y:s.\, v))$;

($\times_0$) $\vdash x:1.\, x =_1 *$;

($\times_1$) $\vdash x:A, y:B.\, \mathrm{fst}(\langle x, y\rangle) =_A x$;

($\times_2$) $\vdash x:A, y:B.\, \mathrm{snd}(\langle x, y\rangle) =_B y$;

($\times_3$) $\vdash z:A\times B.\, \langle \mathrm{fst}(z), \mathrm{snd}(z)\rangle =_{A\times B} z$;

($+_0$) $\vdash \vec{x}:\vec{A}.\, \mathrm{when}(\mathrm{inl}_B(a), t, s) =_C t\cdot a$;

($+_1$) $\vdash \vec{x}:\vec{A}.\, \mathrm{when}(\mathrm{inr}_D(b), t, s) =_C s\cdot b$;

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The λ-calculus

$\hookrightarrow$ (Lambda calculus)

$(+_2)$ when $y : A_1 \not\in \mathrm{FV}(x_1 : A_1 + A_2) \cup \mathrm{FV}(x_3 : B_1 \to C) \cup \mathrm{FV}(x_4 : B_2 \to C)$ and $y : A_2 \not\in \mathrm{FV}(x_2 : A_1 + A_2) \cup \mathrm{FV}(x_3 : B_1 \to C) \cup \mathrm{FV}(x_4 : B_2 \to C)$

$$\begin{aligned}
&\vdash x_0 : A_1 + A_2, x_1 : A_1 \to (B_1 + B_2), x_2 : A_2 \to (B_1 + B_2),\\
&\ x_3 : B_1 \to C, x_4 : B_2 \to C.\\
&\ \mathrm{when}(\mathrm{when}(x_0, x_1, x_2), x_3, x_4) =_C\\
&\ =_C \mathrm{when}(x_0, (\lambda y : A_1.\, \mathrm{when}(x_1 \cdot y, x_3, x_4)),\\
&\qquad\qquad (\lambda y : A_2.\, \mathrm{when}(x_2 \cdot y, x_3, x_4))) \ ;
\end{aligned}$$

$(+_3)$ $\vdash x : A, y : 0.\ \mathrm{F}_A \cdot y =_A x$;

$(\to_0)$ $\vdash \vec{x} : \vec{A}.\ (\lambda y : C.\, s) \cdot t =_B s[t/y]$;

$(\to_1)$ $\vdash \vec{x} : \vec{A}.\ (\lambda y : C.\, t \cdot y) =_{C \to B} t$ where $y : C \not\in \mathrm{FV}(t : C \to B)$;

$(\forall_0)$ $\vdash \vec{x} : \vec{A}.\ \mathrm{allE}(\mathrm{allI}(\lambda z : s.\, t), r) =_{B[r/z]} t[r/z]$;

$(\forall_1)$ $\{\vec{x} : \vec{A}.\ \mathrm{allE}(u, r) =_B \mathrm{allE}(v, r)\}_{r : s \in \mathrm{LTerms}(\Sigma)} \vdash \vec{x} : \vec{A}.\ u =_{(\forall z : s.\, B)} v$;

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The λ-calculus

$\hookrightarrow$ (Lambda calculus)

($\exists_0$) $\vdash \vec{x} : \vec{A}.\ \mathrm{exE}(\mathrm{exI}_z(t), (\lambda z : s.\ v)) =_B (v[r/z]) \cdot t$;

($\exists_1$) $\vec{x} : \vec{A}.\ \mathrm{exE}(u, (\lambda z : s.\ r)) =_B \mathrm{exE}(u, (\lambda z : s.\ t)) \vdash \vec{x} : \vec{A}.\ r =_{C \to B} t$ where $\mathrm{FV}(r : C \to B) = \mathrm{FV}(t : C \to B)$;

($\exists_2$) $\vdash v : (\exists y : s.\ A).\ w =_B \mathrm{exE}(v, (\lambda y : s.\ (\lambda z : A.\ w[\mathrm{exI}_y(z)/v])))$ with $z : A \notin \mathrm{FV}(w : B)$;

($\exists_3$) $\vdash \vec{x} : \vec{A}.\ \mathrm{exE}(\mathrm{exE}(a, (\lambda y : s.\ (\lambda z : D.\ b))), (\lambda y : s.\ c)) =_C$
$=_C \mathrm{exE}(a, (\lambda y : s.\ (\lambda z : D.\ \mathrm{exE}(b, (\lambda y : s.\ c)))))$;

($\exists_4$) $\vdash \vec{x} : \vec{A}.\ \mathrm{exE}(a, (\lambda y : s.\ (\lambda z : C.\ b[\mathrm{exI}_y(z)/w]))) =_B b[a/w]$ with $z : C \notin \mathrm{FV}(b : B)$.

Semantics

Definition 5 (Logically distributive category)

Fixed a λ-signature $\Sigma = \langle S, F, R, \mathrm{Ax} \rangle$, a category $\mathbb{C}$ together with a map $M\colon \lambda\mathrm{Types}(\Sigma) \to \mathrm{Obj}\,\mathbb{C}$ is said to be *logically distributive* if it satisfies the following seven conditions:

1. $\mathbb{C}$ has finite products;
2. $\mathbb{C}$ has finite co-products;
3. $\mathbb{C}$ has exponentiation;
4. $\mathbb{C}$ is *distributive*, i.e., for every $A, B, C \in \mathrm{Obj}\,\mathbb{C}$, the arrow $\Delta = [1_A \times \iota_1, 1_A \times \iota_2]\colon (A \times B) + (A \times C) \to A \times (B + C)$ has an inverse, where $[\_,\_]$ is the co-universal arrow of the $(A \times B) + (A \times C)$ co-product, $\_ \times \_$ is the product arrow, 1_A is the identity arrow on A, and $\iota_1\colon B \to B + C$, $\iota_2\colon C \to B + C$ are the canonical injections of the $B + C$ co-product.

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Semantics

$\hookrightarrow$ (Logically distributive category)

For every $s \in S$, $A \in \lambda\mathrm{Types}(\Sigma)$, and $x \in V_s$, let $\Sigma_A(x:s)\colon \mathrm{LTerms}(\Sigma)(s) \to \mathbb{C}$ be the functor from the discrete category $\mathrm{LTerms}(\Sigma)(s) = \{t:s \mid t:s \in \mathrm{LTerms}(\Sigma)\}$ to $\mathbb{C}$ defined by $t:s \mapsto M(A[t/x])$.

Also, for every $s \in S$, $A \in \lambda\mathrm{Types}(\Sigma)$, and $x \in V_s$, let $\mathbb{C}_{(\forall x:s.A)}$ be the subcategory of $\mathbb{C}$ whose objects are the vertices of the cones on $\Sigma_A(x:s)$ such that they are of the form MB for some $B \in \lambda\mathrm{Types}(\Sigma)$ and $x:s \notin \mathrm{FV}(B)$. Moreover, the arrows of $\mathbb{C}_{(\forall x:s.A)}$, apart identities, are the arrows in the category of cones over $\Sigma_A(x:s)$ having the objects of $\mathbb{C}_{(\forall x:s.A)}$ as domain and $M(\forall x:s.A)$ as co-domain. $\hookrightarrow$

Semantics

$\hookrightarrow$ (Logically distributive category)

Finally, for every $s \in S$, $A \in \lambda\text{Types}(\Sigma)$, and $x \in V_s$, let $\mathbb{C}_{(\exists x:s.A)}$ be the subcategory of $\mathbb{C}$ whose objects are the vertices of the co-cones on $\Sigma_A(x:s)$ such that they are of the form MB for some $B \in \lambda\text{Types}(\Sigma)$ and $x:s \not\in \text{FV}(B)$. Moreover, the arrows of $\mathbb{C}_{(\exists x:s.A)}$, apart identities, are the arrows in the category of co-cones over $\Sigma_A(x:s)$ having the objects of $\mathbb{C}_{(\exists x:s.A)}$ as co-domain and $M(\exists x:s.A)$ as domain.

5. All the subcategories $\mathbb{C}_{(\forall x:s.A)}$ have terminal objects, and all the subcategories $\mathbb{C}_{(\exists x:s.A)}$ have initial objects;

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Semantics

$\hookrightarrow$ (Logically distributive category)

6. The M map is such that
   6.1 $M(0) = 0$, the initial object of $\mathbb{C}$;
   6.2 $M(1) = 1$, the terminal object of $\mathbb{C}$;
   6.3 $M(A \times B) = MA \times MB$, the binary product in $\mathbb{C}$;
   6.4 $M(A + B) = MA + MB$, the binary co-prooduct in $\mathbb{C}$;
   6.5 $M(A \to B) = MB^{MA}$, the exponential object in $\mathbb{C}$;
   6.6 $M(\forall x : s. A)$ is the terminal object in the subcategory $\mathbb{C}_{(\forall x : s. A)}$;
   6.7 $M(\exists x : s. A)$ is the initial object in the subcategory $\mathbb{C}_{(\exists x : s. A)}$;
7. For every $x \in V_s$, $A, B \in \lambda\mathrm{Types}(\Sigma)$ with $x : s \not\in \mathrm{FV}(A)$, $MA \times M(\exists x : s. B)$ is an object of $\mathbb{C}_{(\exists x : s. A \times B)}$ since, if $\left(M(\exists x : s. B), \{\delta_t\}_{t : s \in \mathrm{LTerms}(\Sigma)}\right)$ is a co-cone over $\Sigma_B(x : s)$, and there is one by condition (5), then $\left(MA \times M(\exists x : s. B), \{1_{MA} \times \delta_t\}_{t : s \in \mathrm{LTerms}(\Sigma)}\right)$ is a co-cone over $\Sigma_{A \times B}(x : s)$. Thus, there is a unique arrow $! : M(\exists x : s. A \times B) \to MA \times M(\exists x : s. B)$ in $\mathbb{C}_{(\exists x : s. A \times B)}$. Our last condition requires that the arrow ! has an inverse.

Semantics

Definition 6 (Σ-structure)

Given a λ-signature $\Sigma = \langle S, F, R, \mathrm{Ax} \rangle$, a Σ-structure is a triple $\langle \mathbb{C}, M, M_{\mathrm{Ax}} \rangle$ such that $\mathbb{C}$ together with M forms a logically distributive category and M_{Ax} is a map from Ax such that $M_{\mathrm{Ax}}(a : A \to B) \in \mathrm{Hom}_{\mathbb{C}}(MA, MB)$.

Semantics

Definition 7 (λ-terms semantics)

Fixed a Σ-structure $\langle \mathbb{C}, M, M_{\mathrm{Ax}} \rangle$, let $A \equiv A_1 \times \cdots A_n$, and let $\vec{x} \equiv x_1 : A_1, \ldots, x_n : A_n$ be a context. The *semantics of a term-in-context* $\vec{x}.\, t : B$, notation $[\![\vec{x}.\, t : B]\!]$, is an arrow in $\mathrm{Hom}_{\mathbb{C}}(MA, MB)$ inductively defined as follows:

1. $[\![\vec{x}.\, x_i : A_i]\!] = \pi_i$, the i-th projector of the product $MA = MA_1 \times \cdots \times MA_n$;
2. if $a : C \to B \in \mathrm{Ax}$ then $[\![\vec{x}.\, a(t) : B]\!] = M_{\mathrm{Ax}} a \circ [\![\vec{x}.\, t : C]\!]$;
3. $[\![\vec{x}.\, \langle s, t \rangle : B \times C]\!] = ([\![\vec{x}.\, s : B]\!], [\![\vec{x}.\, t : C]\!])$ where $(\_, \_)$ is the universal arrow of the product $MB \times MC$;
4. $[\![\vec{x}.\, \mathrm{fst}(t) : B]\!] = \pi_1 \circ [\![\vec{x}.\, t : B \times C]\!]$ where π_1 is the first canonical projector of the product $MA \times MB$;
5. $[\![\vec{x}.\, \mathrm{snd}(t) : C]\!] = \pi_2 \circ [\![\vec{x}.\, t : B \times C]\!]$ where π_2 is the second canonical projector of the product $MA \times MB$;

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Semantics

$\hookrightarrow$ (λ-terms semantics)

6. $[\![\vec{x}.\,(\lambda z : C.\,t) : C \to B]\!]$ is the exponential transpose of $[\![\vec{x}, z : C.\,t : B]\!] \colon MA \times MC \to MB$;
7. $[\![\vec{x}.\,s \cdot t : B]\!] = \mathrm{ev} \circ ([\![\vec{x}.\,s : C \to B]\!], [\![\vec{x}.\,t : C]\!])$ where ev is the exponential evaluation arrow;
8. $[\![\vec{x}.\,\mathrm{inl}_B(t) : C + B]\!] = \iota_1 \circ [\![\vec{x}.\,t : C]\!]$ with ι_1 the first canonical injection of the co-product $MC + MB$;
9. $[\![\vec{x}.\,\mathrm{inr}_C(t) : C + B]\!] = \iota_2 \circ [\![\vec{x}.\,t : B]\!]$ with ι_2 the second canonical injection of the co-product $MC + MB$;

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Semantics

$\hookrightarrow$ (λ-terms semantics)

10. calling $[\_,\_]$ the co-universal arrow of $(MA \times MC_1) + (MA \times MC_2)$, $(\_,\_)$ the universal arrow of $MA \times (MC_1 + MC_2)$, and noticing that the arrow $\Delta\colon (MA \times MC_1) + (MA \times MC_2) \to MA \times (MC_1 + MC_2)$ has an inverse because $\mathbb{C}$ with M is logically distributive

$$\begin{aligned}\llbracket \vec{x}.\, \mathrm{when}(t,u,v) : B \rrbracket = [&\mathrm{ev} \circ (\llbracket \vec{x}.\, u : C_1 \to B \rrbracket \times 1_{MC_1}), \\ &\mathrm{ev} \circ (\llbracket \vec{x}.\, v : C_2 \to B \rrbracket \times 1_{MC_2})] \circ \\ &\circ \Delta^{-1} \circ (1_{MA}, \llbracket \vec{x}.\, t : C_1 + C_2 \rrbracket) \quad ;\end{aligned}$$

11. $\llbracket \vec{x}.\, * : 1 \rrbracket = ! : MA \to 1$, the universal arrow of the terminal object;
12. $\llbracket \vec{x}.\, \mathrm{F}_B : 0 \to B \rrbracket$ is the exponential transpose of $(! : 0 \to MB) \circ (\pi_{n+1} : MA \times 0 \to 0)$;

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Semantics

$\hookrightarrow$ (λ-terms semantics)

13. $[\![\vec{x}.\, \mathsf{allI}(\lambda z : s.\, t) : (\forall z : s.\, B)]\!] = \beta \circ \alpha$ where $\alpha \equiv 1_{MA_{i_1}} \times \cdots \times 1_{MA_{i_k}} : MA \to MA'$ with $A' \equiv A_{i_1} \times \cdots \times A_{i_k}$, where $\vec{x}' \equiv \{x_{i_1} : A_{i_1}, \ldots, x_{i_k} : A_{i_k}\} = \mathrm{FV}(t : B)$, and $\beta : MA' \to M(\forall z : s.\, B)$ is the universal arrow from MA' to the terminal object in $\mathbb{C}_{\forall z : s.\, B}$;
14. $[\![\vec{x}.\, \mathsf{allE}(t, r) : B[r/z]]\!] = p_r \circ [\![\vec{x}.\, t : (\forall z : s.\, B)]\!]$ where $p_r : M(\forall z : s.\, B) \to M(B[r/z])$ is the r-th projector of the unique cone on $\Sigma_B(z : s)$ whose vertex is $M(\forall z : s.\, B)$.
    It is worth noticing that $p_r = [\![w : (\forall z : s.\, B).\, \mathsf{allE}(w, r) : B[r/z]]\!]$;
15. $[\![\vec{x}.\, \mathsf{exI}_z(t) : (\exists z : s.\, B)]\!] = j_r \circ [\![\vec{x}.\, t : B[r/z]]\!]$ where $j_r : M(B[r/z]) \to M(\exists z : s.\, B)$ is the r-th injection of the unique co-cone on $\Sigma_B(z : s)$ whose vertex is $M(\exists z : s.\, B)$.
    It is worth noticing that $j_r = [\![w : B[r/z].\, \mathsf{exI}_z(w) : (\exists z : s.\, B)]\!]$;

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Semantics

$\hookrightarrow$ (λ-terms semantics)

16. $[\![\vec{x}.\, \mathrm{exE}(t, (\lambda z : s.\, r)) : B]\!] = \gamma \circ \beta^{-1} \circ (\alpha, [\![\vec{x}.\, t : (\exists z : s.\, C)]\!])$ where
    - 16.1 $\alpha \equiv 1_{MA_{i_1}} \times \cdots \times 1_{MA_{i_k}} : MA \to MA'$ with $A' \equiv A_{i_1} \times \cdots \times A_{i_k}$, where $\vec{x}' \equiv \{x_{i_1} : A_{i_1}, \ldots, x_{i_k} : A_{i_k}\} = \mathrm{FV}(t : (\exists z : s.\, C)) \cup \mathrm{FV}(r : C \to B)$;
    - 16.2 $\beta : M(\exists z : s.\, A' \times C) \to MA' \times M(\exists z : s.\, C)$ is the co-universal arrow in the subcategory $\mathbb{C}_{\exists z : s.\, A' \times C}$;
    - 16.3 $\gamma : M(\exists z : s.\, A' \times C) \to MB$ is the co-universal arrow in $\mathbb{C}_{\exists z : s.\, A' \times C}$.

Soundness

Definition 8 (Validity)

An equality-in-context $\vec{x}.\, s =_A t$ is *valid* in the λ*-theory* T, a set of equalities-in-context, when, in every logically distributive category $\mathbb{C}$, each model M of T is also a model of $\vec{x}.\, s =_A t$.

A Σ-structure M in $\mathbb{C}$ is a *model of a theory* T when it is a model of each ϕ in T.

Finally, M is a *model of an equality-in-context* $\vec{x}.\, t =_A s$ if $[\![\vec{x}.\, t : A]\!] = [\![\vec{x}.\, s : A]\!]$.

Theorem 9 (Soundness)

If an equation-in-context $\vec{x}.\, s =_A t$ is derivable from a λ-theory T, then $\vec{x}.\, s =_A t$ is valid in each model of T in every logically distributive category.

Completeness

Definition 10 (Syntactical equivalence)

Given a λ-theory T, the *syntactical equivalence* of two terms-in-context is defined by fixing the generated equivalence classes. Precisely, the equivalence class $[x : A.\, t : B]$ is defined as the minimal set, composed by terms-in-context, such that

1. $x : A.\, t : B \in [x : A.\, t : B]$—reflexivity;
2. if $T \vdash \vec{y} : \vec{D}.\, s =_C r$, where $\vec{y} : \vec{D}.\, s =_C r$ is an equality-in-context, and $\vec{y} : \vec{D}.\, s : C \in [x : A.\, t : B]$, then $\vec{y} : \vec{D}.\, r : C \in [x : A.\, t : B]$—closure under provable equivalence;

$\hookrightarrow$

Completeness

↪ (Syntactical equivalence)

3. if $\vec{y} : \vec{D}.\, s : C$ is a term-in-context and, for some $1 \leq i < m$ and $z : D_i \times D_{i+1} \notin \mathrm{FV}(s : C) \cup \{y_1 : D_1, \dots, y_m : D_m\}$, it happens that

$$y_1 : D_1, \dots, y_{i-1} : D_{i-1}, z : D_i \times D_{i+1}, y_{i+1} : D_{i+2}, \\ \dots, y_m : D_m.\, s[\mathrm{fst}(z)/y_i][\mathrm{snd}(z)/y_{i+1}] : C \in [x : A.\, t : B] \enspace ,$$

then $\vec{y} : \vec{D}.\, s : C \in [x : A.\, t : B]$—closure under associativity in contexts;

4. if $\vec{y} : \vec{D}.\, s : C$ is a term-in-context and, for some $1 \leq i < m$ and $z : D_{i+1} \times D_i \notin \mathrm{FV}(s : C) \cup \{y_1 : D_1, \dots, y_m : D_m\}$, it happens that

$$y_1 : D_1, \dots, y_{i-1} : D_{i-1}, z : D_{i+1} \times D_i, y_{i+1} : D_{i+2}, \\ \dots, y_m : D_m.\, s[\mathrm{snd}(z)/y_i][\mathrm{fst}(z)/y_{i+1}] : C \in [x : A.\, t : B] \enspace ,$$

then $\vec{y} : \vec{D}.\, s : C \in [x : A.\, t : B]$—closure under commutativity in contexts;

↪

Completeness

$\hookrightarrow$ (Syntactical equivalence)

5. if $\vec{y}:\vec{D}.s:C \in [x:A.y:B]$ and $z:D_i \notin \mathrm{FV}(s:C) \cup \{y_1:D_1,\ldots,y_m:D_m\}$ for some $1 \leq i \leq m$, then

$$y_1:D_1,\ldots,y_{i-1}:D_{i-1},z:D_i,y_{i+1}:D_{i+1},\ldots,y_m:D_m.s[z/y_i]:C$$

   is in $[x:A.t:B]$—closure under α-renaming in contexts.

Completeness

Definition 11 (Syntactical category)

Given a λ-theory T, the *syntactical category* $\mathbb{C}_T$ has $\lambda\text{Types}(\Sigma)$ as objects, where Σ is the λ-signature of T, and the equivalence classes $[x:A.\,t:B]\colon A \to B$ as arrows.
Identities are given by the classes $[x:A.\,x:A]\colon A \to A$ for each λ-type A, and composition is given by substitution:

$$[y:B.\,s:C] \circ [x:A.\,t:B] = [x:A.\,s[t/y]:C] \ .$$

Moreover, the map $M_T\colon \lambda\text{Types}(\Sigma) \to \text{Obj}\,\mathbb{C}_T$ is defined as $M_T A = A$.

Completeness

Proposition 12

The $\mathbb{C}_T$ category is logically distributive.

Proposition 13

Given a λ-theory T on the Σ signature, the Σ-structure $\langle \mathbb{C}_T, M_T, M_{\mathsf{Ax}} \rangle$ on the corresponding syntactical category is defined by M_{Ax} which maps $f\colon A \to B \in \mathsf{Ax}$ to $[x : A.\, f(x) : B]$.
This Σ-structure is a model for T and, moreover, it satisfies exactly those equalities-in-context which are provable in T.

Completeness

Proposition 14

For every logically distributive category $\mathbb{C}$, there is a biijection between equivalence classes, modulo natural equivalences, of structure-preserving functors $\mathbb{C}_T \to \mathbb{C}$ and equivalence classes, modulo isomorphisms, of models of T in $\mathbb{C}$, induced by the map $F \mapsto F(M_T)$.

Theorem 15 (Completeness)

If $\vec{x}.\, s =_A t$ is an equality-in-context valid in every model for T in each logically distributive category, then $T \vdash \vec{x}.\, s =_A t$.

Soundness and Completeness in Logic

Definition 16 (Valid type)

A λ-type A is *valid in the model* $\mathscr{N} = \langle \mathbb{N}, N, N_{\mathrm{Ax}} \rangle$ when there exists an arrow $1 \to NA$ in $\mathbb{N}$.

A λ-type A is a *logical consequence in the model* $\mathscr{N}$ of the λ-types $B_1, \ldots, B_n$ when there exists $N\big(B_1 \times \cdots \times B_n\big) \to NA$ in $\mathbb{N}$.

A λ-type A is a *logical consequence* of $B_1, \ldots, B_n$ when it is a logical consequence of $B_1, \ldots, B_n$ in every model in every logically distributive category.

Soundness and Completeness in Logic

Proposition 17

A λ-type A is a logical consequence of $B_1, \ldots, B_n$ if and only if there exists a term-in-context $x : B_1 \times \cdots \times B_n . t : A$.

Corollary 18

A λ-type A is a logical consequence of $B_1, \ldots, B_n$ if and only if there is a proof of A from the hypotheses $B_1, \ldots, B_n$, when λ-types are interpreted as logical formulae and λ-terms as logical proofs, according to the Curry-Howard isomorphism.

The end

Questions?