Tactics for Translation of Tableau in Natural Deduction

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Abstract. In this paper we show an algorithm to translate intuitionistic tableau proofs into natural deduction-like proofs. The resulting algorithm is proved correct and is used to validate the answers of an intuitionistic tableau prover developed inside the logical framework Isabelle. We think that our contribution is interesting since it presents an efficient way to certify the correctness of answers from a not formally verified theorem prover.

1 Introduction

In this paper we present a method to convert tableau proofs into natural-like derivations. We will show an abstract algorithm to translate closed tableaux for intuitionistic first-order predicate logic into Prawitz's natural deduction calculus [15] for the same logic. Originally, this problem was posed in [19] by L. Wos for resolution-based theorem provers. As P. Andrews said in [1], "substantial work has been done not only on such translations, but also on improving the structure of natural deduction proofs and translating them into natural language". Other approaches which could resemble ours, but are different in the choice of the starting calculus, are described in [4, 14].

The distinctive character of our approach lies in the way we use the translation algorithm. Essentially, we do not require that the starting tableau calculus is sound. This fact allows to use the translation algorithm to certify the answers from an automatic tableau prover which can develop proofs in a non-sound calculus, either because of an error, or on purpose, to improve performances.

Let us suppose to have an environment E where we can develop proofs step by step in natural deduction, and let us suppose that this environment is correct. Now, let P be an automatic theorem prover, which, given a formula ϕ, returns "False" if it is not able to prove ϕ, and "True" plus a tableau T for ϕ if it is able to construct T. We do not assume that P is correct, i.e., that whenever P returns "True" and T, ϕ is a theorem and T is a proof for ϕ. If $\textsc{Tr}$ is a translation algorithm, because it is formally correct, $\textsc{Tr}(T)$ is a proof in E if and only if T is correct, i.e., if ϕ is a theorem. In this way the system $E + P + \textsc{Tr}$ provides trustable answers, even if P may be non-correct. Later, we will show that it makes sense to use provers which are not correct, so we can avoid significant computational cost without restrict too much the set of provable theorems.

In practice, we developed a package for the ISABELLE logical framework [13], where an automatic proving procedure for intuitionistic first-order logic is used to (partially) decide the validity of goals; an implementation of the translation algorithm is coupled to that prover in order to guarantee that every positive answer is trustable. The whole package is available in beta version at **ftp://dotto.usr.dsi.unimi.it logic/tableau.tar.gz**.

In the real implementation, the translation algorithm construct a *tactic*, i.e., a function which sequentially applies elementary inference rules; in this way, even a non-closed tableau can be converted in a partial proof. The correctness of this variant of the translation algorithm follows from the propositions we will derive in Section 3. As a rough measure, the implementation we developed, computes a natural deduction proof from a given tableau in $O(n^3)$ steps, where n is the number of node in the tableau.

We choose the tableau calculus developed in [12]. Our choice is motivated by the problem of simplifying the search for proofs by reducing the amount of *duplications*, where a duplication occur in a proof of a tableau calculus whenever a formula already used by an inference rule is used again by the same rule (for a comprehensive discussion about duplications, see [12]). A quite similar problem has been taken into account on the side of sequent calculi, where the counterpart of duplication is the *elimination of contractions*. There, in the intuitionistic framework important results have been obtained by Dyckhoff [5] and, independently, Hudelmaier [8], who have exhibited cut-free and *contraction* free sequent calculi for Intuitionistic Propositional Logic, where a sequent calculus is contraction-free if no formula occurring in the lower sequent of an inference rule (i.e., the sequent obtained by applying the rule) can occur in some of the upper sequents (i.e., the sequents to which the rule is applied); in [5] also a contraction-free natural calculus is given while in [8] it is shown that the involved calculus gives rise to an $O(n \log n)$-SPACE decision procedure for Intuitionistic Propositional Logic. The tableau calculus of [12] (which is a refinement of those in [10, 11]) has improved Fitting's tableau calculus for Intuitionistic Predicate Logic [7] by reducing the amount of duplications involved in its proofs. To do so, in [12] the $\mathbf{F_c}$-signed formulas have been introduced near the $\mathbf{F}$-signed and the $\mathbf{T}$-signed formulas of Fitting's calculus (see the discussion in [10]), and the rules for implication have been refined, taking into account the ideas of Dyckhoff's paper. Thus, the tableau calculus for Intuitionistic Predicate Logic in [12], is nearly optimal from the point of view of the elimination of duplications (such a calculus completely avoids duplications in the propositional framework, thus providing, for tableau calculi, a result comparable with Dyckhoff's and Hudelmaier's one for sequent calculi; on the other hand, at the predicate level, not all duplications, or *mutatis mutandis*, not all contractions, can be cut off).

The paper has the following structure: the next section contains the set of definitions and results we need in order to introduce our translation algorithm. Section 3 shows the translation algorithm and proves its correctness. In the last part we draw some conclusions, expanding the discussion above on correctness of a theorem prover.

2 Preliminary Definitions

The set of *well formed formulas* (*wff's* for short) is defined, as usual, starting from the propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, the quantifiers $\forall$ and $\exists$, a denumerable set $\mathcal{V}$ of individual variables, and, for any $n \geq 0$, a denumerable set $\mathcal{P}^n$ of n-ary predicate variables.

IL denotes the set of intuitionistically valid wff's. Our choice for a natural deduction-like system is the calculus **Ni** as shown in [18]; for reference, we report in Table 1 the complete set of inference rules. For the sake of simplicity, we will refer to the calculus with the name **IL**. In [17], **IL** is proven to be sound and complete. A *(predicate) Kripke model* is a quadruple $\underline{K} = \langle P, \leq, \Vdash, \mathcal{D} \rangle$, where

$$\frac{A \quad B}{A \wedge B}\,\wedge\mathrm{I} \qquad \frac{A \wedge B}{A}\,\wedge\mathrm{E}_l \qquad \frac{A \wedge B}{B}\,\wedge\mathrm{E}_r$$

$$\frac{A}{A \vee B}\,\vee\mathrm{I}_l \qquad \frac{B}{A \vee B}\,\vee\mathrm{I}_r \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \prod\\ C\end{array} \quad \begin{array}{c}[B]\\ \prod\\ C\end{array}}{C}\,\vee\mathrm{E}$$

$$\frac{\begin{array}{c}[A]\\ \prod\\ B\end{array}}{A \rightarrow B}\,\rightarrow\mathrm{I} \qquad \frac{A \quad A \rightarrow B}{B}\,\rightarrow\mathrm{E} \qquad \frac{\bot}{A}\,\bot\mathrm{E}$$

$$\frac{A(p)}{\forall\, x.\, A(x)}\,\forall\mathrm{I}\ (*) \qquad \frac{\forall\, x.\, A(x)}{A(t)}\,\forall\mathrm{E}$$

$$\frac{A(t)}{\exists\, x.\, A(x)}\,\exists\mathrm{I} \qquad \frac{\exists\, x.\, A(x) \quad \begin{array}{c}[A(p)]\\ \prod\\ B\end{array}}{B}\,\exists\mathrm{E}\ (**)$$

where, in $(*)$ and $(**)$, p is an eigenvariable.

Table 1. Inference rules for **IL**

$\underline{P} = \langle P, \leq \rangle$ is a partial ordered set, named *poset* or *frame*; $\mathcal{D}$ is the *domain function*, associating, to any element $\alpha \in P$, a domain $\mathcal{D}(\alpha)$, and such that, for any $\alpha, \beta \in P$, if $\alpha \leq \beta$ then $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$; $\Vdash$ is the *forcing relation*, defined between elements of P and atomic wff's with parameters denoting the elements of the domains, and such that, for any atomic wff $p(x_1, \ldots, x_n)$, for any $\alpha, \beta \in P$ and any $a_1, \ldots, a_n \in \mathcal{D}(\alpha)$, if $\alpha \leq \beta$ and $\alpha \Vdash p(a_1, \ldots, a_n)$ then $\beta \Vdash p(a_1, \ldots, a_n)$. The forcing relation is extended to arbitrary closed wff's (with parameters denoting elements of the domains) in the usual way (see [6, 16] for a detailed definition).

The tableau calculi for Intuitionistic Predicate Logic we are going to use is defined in [10–12] and it uses the three signs $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{F_c}$. Given a wff A, a *signed well formed formula* (*swff* for short) will be every expression of the kind $\mathcal{S}A$, where $\mathcal{S} \in \{\mathbf{T}, \mathbf{F}, \mathbf{F_c}\}$.

The meaning of the signs $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{F_c}$ is explained in terms of *realizability* as follows: given a model $\underline{K} = \langle P, \leq, \Vdash, \mathcal{D}\rangle$, an element $\alpha \in P$, and a swff H whose parameters denote elements of $\mathcal{D}(\alpha)$, we say that α *realizes* H *(in* $\underline{K}$*)*, and we write $\alpha \rhd H$ (in $\underline{K}$), if the following conditions hold:

1. If $H \equiv \mathbf{T}A$, then $\alpha \Vdash A$;
2. If $H \equiv \mathbf{F}A$, then $\alpha \nVdash A$;
3. If $H \equiv \mathbf{F_c}A$, then $\alpha \Vdash \neg A$.

We say that α *realizes a set of swff's* S (and we write $\alpha \rhd S$) iff α realizes every swff in S. A set of swff's S is *realizable* iff there is some element α of a Kripke model $\underline{K}$ such that $\alpha \rhd S$. By a *configuration* we mean a finite sequence $S_1|S_2|\ldots|S_n$ (with $n \geq 1$), where every S_j is a set of swff's. A set of swff's of a configuration is called *node*. A *configuration is realizable* iff at least a S_j is realizable.

The intuitionistic tableau calculus **IL**-T consists of the rules in Table 2, where S_c is the *certain part of* S; formally: $S_c = \{\mathbf{T}X \mid \mathbf{T}X \in S\} \cup \{\mathbf{F_c}X \mid \mathbf{F_c}X \in S\}$.

Definition 1. *Let S be a set of swff's, a* tableau *(sometimes called* **IL**-proof-table*) for S is a finite sequence of configurations $C_1, \ldots, C_n$ such that*

- *$C_1 = \{S\}$, and*
- *$C_{i+1} = \{S_1, \ldots, S_n\} \cup \{N_1, \ldots, N_k\}$, where $C_i = \{S_1, \ldots, S_n\} \cup \{M\}$ and*

$$\frac{M}{N_1 \mid \ldots \mid N_k}$$

is an instance of an expansion rule as in Table 2.

The active *wff in a node is the swff an expansion rule is applied to, the other swff's in that node are called* context*. A node in the last configuration (the* terminal configuration*) of a tableau is called a* terminal node*.*

A **IL**-proof-table is *closed* iff all the sets S_j of its final configuration are contradictory, where a set S is *contradictory* if (for some A) either $\{\mathbf{T}A, \mathbf{F}A\} \subseteq S$ or $\{\mathbf{T}A, \mathbf{F_c}A\} \subseteq S$. Accordingly, we call *complementary pair* any set of swff's of the form $\{\mathbf{T}A, \mathbf{F}A\}$ or of the form $\{\mathbf{T}A, \mathbf{F_c}A\}$. It is immediate to verify that if S is a set of swff's and contains a complementary pair, then S is not realizable. A *proof* of a wff B in **IL** is a closed **IL**-proof-table starting from $\{\mathbf{F}B\}$. Finally, we say that a set of swff's S is **IL**-*consistent* iff no **IL**-proof-table starting from S is closed.

Every rule of **IL**-T tableau calculus is applied to a swff of a set S_i occurring in a configuration $S_1 \mid \cdots \mid S_i \mid \ldots$; e.g., the notation $S, \mathbf{T}(A \wedge B)$ points out that the rule $\mathbf{T}\wedge$ is applied to the swff $\mathbf{T}(A \wedge B)$ of the set $S \cup \{\mathbf{T}(A \wedge B)\}$, where S is possibly empty. We remark that all the rules of the calculus **IL**-T, except for $\mathbf{T}\forall$, $\mathbf{F}\exists$, $\mathbf{F_c}\exists$, $\mathbf{F_c}\forall$ and $\mathbf{T}\!\rightarrow\!\forall$, are duplication-free, in the sense explained in [2, 3, 5, 10–12].

In [10–12], it is proved that **IL**-T is sound and complete according to the standard intuitionistic semantics based on Kripke models.

$\mathbf{T}$ – rules	$\mathbf{F}$ – rules	$\mathbf{F_c}$ – rules
$\dfrac{S, \mathbf{T}(A \wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\ \mathbf{T}\wedge$	$\dfrac{S, \mathbf{F}(A \wedge B)}{S, \mathbf{F}A \mid S, \mathbf{F}B}\ \mathbf{F}\wedge$	$\dfrac{S, \mathbf{F_c}(A \wedge B)}{S_c, \mathbf{F_c}A \mid S_c, \mathbf{F_c}B}\ \mathbf{F_c}\wedge$
$\dfrac{S, \mathbf{T}(A \vee B)}{S, \mathbf{T}A \mid S, \mathbf{T}B}\ \mathbf{T}\vee$	$\dfrac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A, \mathbf{F}B}\ \mathbf{F}\vee$	$\dfrac{S, \mathbf{F_c}(A \vee B)}{S, \mathbf{F_c}A, \mathbf{F_c}B}\ \mathbf{F_c}\vee$
See special table for $\mathbf{T}\!\rightarrow$	$\dfrac{S, \mathbf{F}(A \rightarrow B)}{S_c, \mathbf{T}A, \mathbf{F}B}\ \mathbf{F}\!\rightarrow$	$\dfrac{S, \mathbf{F_c}(A \rightarrow B)}{S_c, \mathbf{T}A, \mathbf{F_c}B}\ \mathbf{F_c}\!\rightarrow$
$\dfrac{S, \mathbf{T}(\neg A)}{S, \mathbf{F_c}A}\ \mathbf{T}\neg$	$\dfrac{S, \mathbf{F}(\neg A)}{S_c, \mathbf{T}A}\ \mathbf{F}\neg$	$\dfrac{S, \mathbf{F_c}(\neg A)}{S_c, \mathbf{T}A}\ \mathbf{F_c}\neg$
$\dfrac{S, \mathbf{T}(\forall\, x.\, A(x))}{S, \mathbf{T}A(a), \mathbf{T}(\forall\, x.\, A(x))}\ \mathbf{T}\forall$	$\dfrac{S, \mathbf{F}(\forall\, x.\, A(x))}{S_c, \mathbf{F}A(a)}\ \mathbf{F}\forall$ with a new	$\dfrac{S, \mathbf{F_c}(\forall\, x.\, A(x))}{S_c, \mathbf{F_c}A(a), \mathbf{F_c}(\forall\, x.\, A(x))}\ \mathbf{F_c}\forall$ with a new
$\dfrac{S, \mathbf{T}(\exists\, x.\, A(x))}{S, \mathbf{T}A(a)}\ \mathbf{T}\exists$ with a new	$\dfrac{S, \mathbf{F}(\exists\, x.\, A(x))}{S, \mathbf{F}A(a)}\ \mathbf{F}\exists$	$\dfrac{S, \mathbf{F_c}(\exists\, x.\, A(x))}{S, \mathbf{F_c}A(a), \mathbf{F_c}(\exists\, x.\, A(x))}\ \mathbf{F_c}\exists$

Rules for $\mathbf{T}\!\rightarrow$	
$\dfrac{S, \mathbf{T}(A \rightarrow B)}{S, \mathbf{F}A \mid S, \mathbf{T}B}\ \mathbf{T}\!\rightarrow_{a\neg}$ with A atomic or negated	$\dfrac{S, \mathbf{T}((A \vee B) \rightarrow C)}{S, \mathbf{T}(A \rightarrow C), \mathbf{T}(B \rightarrow C)}\ \mathbf{T}\!\rightarrow\!\vee$
$\dfrac{S, \mathbf{T}((A \wedge B) \rightarrow C)}{S, \mathbf{T}(A \rightarrow (B \rightarrow C))}\ \mathbf{T}\!\rightarrow\!\wedge$	$\dfrac{S, \mathbf{T}((A \rightarrow B) \rightarrow C)}{S, \mathbf{F}(A \rightarrow B), \mathbf{T}(B \rightarrow C) \mid S, \mathbf{T}C}\ \mathbf{T}\!\rightarrow\rightarrow$
$\dfrac{S, \mathbf{T}((\exists\, x.\, A(x)) \rightarrow B)}{S, \mathbf{T}(\forall\, x.\, (A(x) \rightarrow B))}\ \mathbf{T}\!\rightarrow\!\exists$	$\dfrac{S, \mathbf{T}((\forall\, x.\, A(x)) \rightarrow B)}{S, \mathbf{F}(\forall\, x.\, A(x)), \mathbf{T}((\forall\, x.\, A(x)) \rightarrow B) \mid S, \mathbf{T}B}\ \mathbf{T}\!\rightarrow\!\forall$

Table 2. Expansion rules for intuitionistic tableaux

3 Translating Tableau Proofs into Natural Deduction

In this section we introduce the algorithm which translates tableaux into natural deduction-like proofs, and we prove its correctness. To do so, we define the concept of *proof with gaps*, and we use this notion to define the function Tr, which maps each tableau into a proof which may contain non-specified parts, the gaps. Finally we show how we can remove these gaps when the original tableau is closed.

Definition 2. *The* intuitionistic natural deduction-like calculus with gaps **ILG** *is defined from the inference rules in Table 1 plus*

$$\frac{\gamma_1 \cdots \gamma_n}{\alpha}\,\mathrm{G} \quad .$$

A proof in this system is a generalization of proofs in **IL** calculus; for this reason we will refer to deductions in this system as *proofs with gaps* (Pg) in **IL**.

To indicate gaps into a Pg-proof, we use the notation

$$\Xi(G_1, \ldots , G_n)$$

where $G_1, \ldots , G_n$ are the application of the G rule (the gaps, for short) we want to mark. When we write $\Xi(\Pi_1, \ldots , \Pi_n)$ we intend that the gaps $G_1, \ldots ,G_n$ are substituted with the Pg-proofs $\Pi_1, \ldots ,\Pi_n$.

The intuitive meaning of a proof with gaps is that it is a "partial" proof, i.e., a proof where some parts (the gaps) are not yet developed, but their assumptions and conclusions are specified. We use a gap in a proof as a placeholder for another proof, which should be developed according to our translation algorithm.

Definition 3. *The natural deduction-like calculus* **ILGD** *is defined from* **ILG** *plus the rules of Appendix A.*

In Proposition 2, we will prove that the rules of Appendix A can be derived in **ILG**; hence **ILG** and **ILGD** are equivalent, in the sense that they prove the same set of wff's. Again, the notation $\Xi(G_1, \ldots , G_n)$ will indicate that $G_1, \ldots ,$ G_n are the application of the G rule (the gaps, for short) we want to mark in the **ILGD** proof Ξ.

Definition 4. *The translation function* Tr *maps a tableau T into a proof* Tr(T) *in* **ILGD***; the definition is given by induction on the structure of the tableau T:*

– *base case*
  Let $T = \{S\}$ be a tableau consisting of the initial configuration; then

$$\mathrm{Tr}(T) = \frac{\{\phi \mid \mathbf{T}\ \phi \in S\} \cup \{\neg\phi \mid \mathbf{F_c}\ \phi \in S\}}{\bigvee\{\phi \mid \mathbf{F}\ \phi \in S\}}\,\mathrm{G}$$

– *induction case*
Let $C = \{S_1, \ldots, S_n\}$ *be the last configuration of* T*, obtained from a tableau* T' *and its terminal configuration* $C' = \{S'_1, \ldots, S'_m\}$ *by applying a tableau rule to the swff* α *in the node* $S'_j = \Gamma \cup \{\alpha\}$*, and generating as new nodes* $N_1, \ldots, N_k$*.*
Let $\textsc{Tr}(T') = \Xi(G_{S'_j})$*, where* $G_{S'_j}$ *is the gap corresponding to* S'_j*. Then* $\textsc{Tr}(T) = \Xi(\textsc{Tr}(G_{S'_j}))$ *and* $\textsc{Tr}(G_{S'_j})$ *is given in Appendix A, according to the principal connective, the sign of* α*, and the number of* **F***-wff's in* S'_j*.*

We observe that every node in the terminal configuration of T has a corresponding gap in $\textsc{Tr}(T)$, and this correspondence defines what we intend for gap associated with a node.

The translation function $\textsc{Tr}$ is the way how we construct the tactics in Isabelle, in the sense that $\textsc{Tr}(T)$ can be regarded as a sequence of applications of inference rules, and the gaps as intermediate subgoals to be proven. The next propositions and theorem will prove that if T is a closed **IL**-proof-table then we can combine those tactics to construct a proof in **IL**. We remark that, $\textsc{Tr}(T)$ is defined even if the tableau calculus is not sound; in this case we will not be able to derive every translation rule, so we will not obtain a proof in **IL**.

Proposition 1. *Let* T *be a tableau; for any gap* G *in* $\textsc{Tr}(T)$*, there is a node* S *in the terminal configuration of* T *such that* G *is associated with* S *and*

$$G \equiv \frac{\{\phi \mid \mathbf{T}\,\phi \in S\} \cup \{\neg\phi \mid \mathbf{F_c}\,\phi \in S\}}{\bigvee\{\phi \mid \mathbf{F}\,\phi \in S\}}\,\mathrm{G} \quad .$$

Proof. By induction on the structure of the tableau T. The base case, $T = \{S\}$, is trivial. So let's suppose T is derived from T' by expanding a terminal node S' according to the rules in Table 2 and let $\textsc{Tr}(T') = \Xi(G_{S'})$.

By induction hypothesis on T', for every gap G in $\textsc{Tr}(T')$, there exists S, terminal node in T' associated with G, such that

$$G \equiv \frac{\{\phi \mid \mathbf{T}\,\phi \in S\} \cup \{\neg\phi \mid \mathbf{F_c}\,\phi \in S\}}{\bigvee\{\phi \mid \mathbf{F}\,\phi \in S\}}\,\mathrm{G} \quad .$$

By definition, $\textsc{Tr}(T) = \Xi(\textsc{Tr}(G_{S'}))$, i.e., $\textsc{Tr}(T)$ is obtained from $\Xi(G_{S'})$, by replacing the G-rule corresponding to the gap $G_{S'}$ according to the rules of Appendix A. Thus it suffices to prove that the number of Gap introduced by $\textsc{Tr}(\textsc{Tr}(G_{S'}))$ is exactly equal to the splitting introduced by the tableau rule applied to S', and that the new gaps have the required shape.

The proof is by cases according to the rules of Appendix A. For the sake of simplicity we only treat the cases corresponding to the translation rules $\mathbf{T}\wedge$, $\mathbf{F_c}\wedge$, and $\mathbf{F_c} \wedge \bot$; the other cases are treated in the similar way.
• Let $S' = \{\mathbf{T}A_1, \ldots, \mathbf{T}A_n,\ \mathbf{F}B_1, \ldots \mathbf{F}B_m,\ \mathbf{F_c}C_1, \ldots \mathbf{F_c}C_k, \mathbf{T}(H \wedge E)\}$ where the last configuration of T is obtained by expanding the swff $\mathbf{T}(H \wedge E)$ and let

$G_{S'}$ be equal to

$$\dfrac{A_1, \ldots, A_n, \neg C_1, \ldots \neg C_k, (H \wedge E)}{B_1 \vee \ldots \vee B_m}\, G_{S'} \quad .$$

Then $\textsc{Tr}(G_{S'})$, corresponding to the $\mathbf{T}\wedge$ translation rule, is the following:

$$\dfrac{H \wedge E \qquad \dfrac{A_1, \ldots, A_n, \neg C_1, \ldots \neg C_k, H, E}{B_1 \vee \ldots \vee B_m}\,\mathrm{G}^*}{B_1 \vee \ldots \vee B_m}\,\mathbf{T}\wedge$$

On the other hand, the tableau rule $\mathbf{T}\wedge$ transform the set S' into the set $S = \{\mathbf{T}A_1, \ldots, \mathbf{T}A_n, \mathbf{F}B_1, \ldots \mathbf{F}B_m, \mathbf{F_c}C_1, \ldots \mathbf{F_c}C_k, \mathbf{T}H, \mathbf{T}E\}$. Hence, the gap G^* is associated with the set of swff's S and it has the required shape.

• Let $S' = \{\mathbf{T}A_1, \ldots, \mathbf{T}A_n, \mathbf{F}B_1, \ldots \mathbf{F}B_m, \mathbf{F_c}C_1, \ldots \mathbf{F_c}C_k, \mathbf{F_c}(H \wedge E)\}$, $\Gamma = \{A_1, \ldots, A_n\}$, $\Theta = \{\neg C_1, \ldots, \neg C_k\}$, and $D \equiv B_1 \vee \ldots \vee B_m$ where last configuration of T is obtained by expand the swff $\mathbf{F_c}(H \wedge E)$. Moreover, let $G_{S'}$ be equal to

$$\dfrac{\Gamma \cup \Theta \cup \{\neg(H \wedge E)\}}{D}\, G_{S'}$$

Then $\textsc{Tr}(G_{S'})$, corresponding to the $\mathbf{F_c}\wedge$ translation rule or to the $\mathbf{F_c} \wedge \bot$ translation rule respectively are the following:

$$\dfrac{\neg(H \wedge E) \qquad \dfrac{\Gamma \cup \Theta, [H]}{\bot}\,\mathrm{G}_1 \qquad \dfrac{\Gamma \cup \Theta, [E]}{\bot}\,\mathrm{G}_2}{D}\,\mathbf{F_c}\wedge \qquad (\text{if } m \neq 0)$$

$$\dfrac{\neg(H \wedge E) \qquad \dfrac{\Gamma \cup \Theta, [H]}{\bot}\,\mathrm{G}_1 \qquad \dfrac{\Gamma \cup \Theta, [E]}{\bot}\,\mathrm{G}_2}{\bot}\,\mathbf{F_c}\wedge\bot \qquad (\text{if } m = 0)$$

On the other hand, $\mathbf{F_c}\wedge$-rule transform the set S' into $S_1 = \{\mathbf{T}A_1, \ldots, \mathbf{T}A_n, \mathbf{F}B_1, \ldots \mathbf{F}B_m, \mathbf{F_c}C_1, \ldots \mathbf{F_c}C_k, \mathbf{F_c}H\}$ and $S_2 = \{\mathbf{T}A_1, \ldots, \mathbf{T}A_n, \mathbf{F}B_1, \ldots \mathbf{F}B_m, \mathbf{F_c}C_1, \ldots \mathbf{F_c}C_k, \mathbf{F_c}E\}$. Hence, the gaps G_1 and G_2 are associated with the sets of swff's S_1 and S_2, respectively, and they have the required shape. □

Proposition 2. *The rule of Appendix A can be derived in* **ILG**.

Proof. For the sake of simplicity we only treat the cases corresponding to the translation rules $\mathbf{T}\wedge$, $\mathbf{F_c}\wedge$, and $\mathbf{F_c} \wedge \bot$; the other cases are treated in a similar way. Hence,

$$\boxed{\dfrac{A \wedge B \qquad \dfrac{\Gamma, [A], [B]}{D}\,\mathrm{G}}{D}\,\mathbf{T}\wedge \quad \Longrightarrow \quad \dfrac{\Gamma \quad \dfrac{A \wedge B}{A}\,\wedge\mathrm{E} \quad \dfrac{A \wedge B}{B}\,\wedge\mathrm{E}}{D}\,\mathrm{G}}$$

$$\dfrac{\dfrac{\Gamma}{D\vee A}\,\mathrm{G}_1 \qquad \dfrac{\Gamma}{D\vee B}\,\mathrm{G}_2}{D\vee(A\wedge B)}\,\mathbf{F}\wedge$$

$$\Downarrow$$

$$\dfrac{\dfrac{\Gamma}{D\vee A}\,\mathrm{G}_1 \quad \dfrac{[D]_2}{D\vee(A\wedge B)}\,\vee\mathrm{I} \quad \dfrac{\dfrac{\Gamma}{D\vee B}\,\mathrm{G}_2 \quad \dfrac{[D]_1}{D\vee(A\wedge B)}\,\vee\mathrm{I} \quad \dfrac{\dfrac{[A]_2 \quad [B]_1}{A\wedge B}\,\wedge\mathrm{I}}{D\vee(A\wedge B)}\,\vee\mathrm{I}}{D\vee(A\wedge B)}\,\vee\mathrm{E}_1}{D\vee(A\wedge B)}\,\vee\mathrm{E}_2$$

$$\dfrac{\neg(A\wedge B) \quad \dfrac{\Gamma,[\neg A]}{\bot}\,\mathrm{G}_1 \quad \dfrac{\Gamma,[\neg B]}{\bot}\,\mathrm{G}_2}{D}\,\mathbf{F_c}\wedge \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]_1 \quad [B]_2}{A\wedge B}\,\wedge\mathrm{I} \quad \neg(A\wedge B)}{\bot}\,\to\mathrm{E}}{\neg B}\,\to\mathrm{I}_2}{\bot}\,\mathrm{G}_2}{\neg A}\,\to\mathrm{I}_1}{\bot}\,\mathrm{G}_1}{D}\,\bot\mathrm{E}}{}$$

$$\dfrac{\neg(A\wedge B) \quad \dfrac{\Gamma,[\neg A]}{\bot}\,\mathrm{G}_1 \quad \dfrac{\Gamma,[\neg B]}{\bot}\,\mathrm{G}_2}{\bot}\,\mathbf{F_c}\wedge\bot \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[D]_1 \quad [E]_2}{D\wedge E}\,\wedge\mathrm{I} \quad \neg(A\wedge B)}{\bot}\,\to\mathrm{E}}{\neg B}\,\to\mathrm{I}_2}{\bot}\,\mathrm{G}_2}{\neg A}\,\to\mathrm{I}_1}{\bot}\,\mathrm{G}_1$$

□

Using Propositions 1 and 2 one immediately obtains the following corollary.

Corollary 1. *Let T be a tableau; then* $\textsc{Tr}(T)$ *can be translated in a* Pg*-proof.*

In the previous proposition we proved that every rule of Appendix A is derivable in **ILG**. We note that some rules (e.g., $\mathbf{F_c}\wedge\bot$) are instances of others (e.g., $\mathbf{F_c}\wedge$); the reason behind this apparent duplication of rules is that they have a shorter, more "natural" proof. Since the proof of a derived rule becomes part of the proof we generate with TR, it makes sense to have simpler and more natural proofs, so to enhance their readability.

We observe that the way of composing gaps in the expansion of a translation rule may not respect the shape of the rule itself, as in the correctness proof of $\mathbf{F_c}\wedge$. The translation rules are designed in such a way that their applications reflect the content of the corresponding node in the tableau.

Since **ILG** and **ILGD** are essentially the same calculi, then, without loose of generality we can suppose that TR is a function mappings every tableau T into a proof in **ILG**.

Definition 5. *Let T be a tableau and let* $\text{TR}(T) = \Xi(G_{S_1}, \ldots, G_{S_n})$ *where all the gaps belonging to* $\text{TR}(T)$ *are put into evidence. The* Gap closure *of* $\text{TR}(T)$*,* $\Pi_c(\text{TR}(T))$ *is the* PG*-proof obtained from* $\text{TR}(T)$ *by substituting the rules of Appendix A with their derivation according to Proposition 2 and by replacing the gaps* $G_{S_1}, \ldots, G_{S_n}$ *with* PG*-proofs according the following conventions:*

1. *if* S_i *is* **F***-closed, then*

$$G_{S_i} \equiv \frac{A_1, \ldots, A_n, \neg C_1, \ldots \neg C_k, D}{B_1 \vee \ldots \vee B_m \vee D}\,\text{G}$$

   is replaced with

$$\cfrac{\cfrac{D}{B_1 \vee D}\,\vee\text{I}}{B_1 \vee B_2 \vee D}\,\vee\text{I} \atop {\vdots\ \vee\text{I} \atop B_1 \vee \ldots \vee B_m \vee D} \quad \text{(if } m \neq 0\text{)} \qquad\qquad D \quad \text{(if } m = 0\text{)}$$

2. *if* S_i *is* $\mathbf{F_c}$*-closed, then*

$$G_{S_i} \equiv \frac{A_1, \ldots, A_n, \neg C_1, \ldots \neg C_k, D, \neg D}{B_1 \vee \ldots \vee B_m}\,\text{G}$$

   is replaced with

$$\cfrac{\cfrac{D \quad \neg D}{\bot}\,\to\text{E}}{B_1 \vee \ldots \vee B_m}\,\bot\text{E} \quad \text{(if } m \neq 0\text{)} \qquad \cfrac{D \quad \neg D}{\bot}\,\to\text{E} \quad \text{(if } m = 0\text{)}$$

3. *If* S_i *is not closed, then the corresponding gap is left unchanged.*

We observe that closing a gap G in the sense of the previous definition, may cancel antecedents of G which are not essential to complete the proof. It is obvious that Π_c is a function from PG-proofs to PG-proofs. Now, we have to prove that, if T is a closed tableau and P is the result of translating it into **ILG**, then $\Pi_c(P)$ is, indeed, a proof in the **IL** system, i.e., it contains no gaps.

Theorem 1. *Let T be a closed tableau starting from a set S of swff's. Then $\Pi_c(\textsc{Tr}(T))$ is an intuitionistic natural-like proof of*

$$\{\phi \mid \mathbf{T}\,\phi \in S\} \cup \{\neg\phi \mid \mathbf{F_c}\,\phi \in S\} \mathrel{\underset{\mathbf{IL}}{\vdash}} \bigvee\{\phi \mid \mathbf{F}\,\phi \in S\}$$

Proof. By Propositions 1 and 2 , the gaps belonging to the Pg-proof $\textsc{Tr}(T)$ are those associated to the terminal nodes of T. Since T is closed, all gaps in $\textsc{Tr}(T)$ are replaced with proofs without gaps in $\Pi_c(\textsc{Tr}(T))$, hence, $\Pi_c(\textsc{Tr}(T))$ is a Pg-proof without gaps, that is, an intuitionistic natural-like proof. □

4 Conclusions

Summarizing, in this paper we have shown that there is correct algorithm which uniformly maps tableaux into natural-like deductions.

The first consequence of this result, is that, in order to trust the answers of an automatic theorem prover which uses our tableau calculus, we just need to prove that the implementation of the function Tr we developed, is correct. Of course, in this way we get a compromise: we break down the complexity of a formal correctness proof, but we are not able to guarantee the completeness of the prover, that is, the ability to ensure that any theorem will eventually get proven.

The second consequence is more suitable; what Proposition 1 says, is that the translation rules respect the tableau structure. If we chance the starting tableau calculus, it is possible to generate a set of translation rules which permits to prove Proposition 1 in a similar way as we did. In order to prove Theorem 1, it is essential to establish Proposition 2, but this is possible only if the tableau calculus is sound. So, even if the tableau calculus is not sound, we have a translation function which may leave some gaps into the natural-like deduction it generates. It may appear odd to use a non-sound calculus, but it is not. In fact, we are inspecting a possibility which appear very promising: if we let the $\underline{a}$ parameter in the rules $\mathbf{T}\forall$, $\mathbf{F}\exists$ and $\mathbf{F_c}\exists$ to be a free variable which may get instantiated by unification during the process of constructing a tableau, we get a calculus which is not sound, since it is does not handle in the proper way the binding of variables (it is a problem very close to Prolog implementations which avoid the occur-check [9]). This variation of our tableau calculus is much more efficient, since it does not require so many duplications as the sound calculus, but, of course, it may produce closed tableaux which are not **IL**-T proof tables. Our approach is to validate a closed tableau by translating it into an **IL** proof. If this is possible, we have been able to prove in an efficient way a, possibly difficult, theorem; if this is not possible, it means that we applied in a non-sound way an expansion rule, so we are not able to judge whether the goal is a theorem.

The third consequence of our approach depends on the particular shape of the translation rule: since we took care of avoiding unnecessary detours, our natural-like proofs are inspectable. This is important when we want to analyze the strategies the prover adopts.

Concluding, we would like to remark that aur technique can be extended to many other tableau calculi for a great varieties of logic. Moreover, we believe that "validating by translation" could be a winning approach in many situations, where it is important to have only correct answer.

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A Translation Rules

The algorithm which underlies the application of a translation rule is as follows:

1. If S is the context of the node to which a tableau expansion rule is applied, then $\Gamma = \{\phi \mid \mathbf{T}\phi \in S\} \cup \{\neg\phi \mid \mathbf{F_c}\phi \in S\}$ and $D = \bigvee\{\phi \mid \mathbf{F}\phi \in S\}$.
2. In general, a tableau expansion rule is mapped into the translation rule with the same name, unless
   (a) there is no corresponding translation rule (this happens for the $\mathbf{F}\vee$ and the $\mathbf{T}\neg$ rules); in this case, the proof with gaps is left unchanged;
   (b) there is a no$\vee$ or a $\bot$ variant (e.g. $\mathbf{F}\wedge$no$\vee$ and $\mathbf{F_c}\wedge\bot$) and the context of the node to which the tableau rule is applied contains no **F**-wffs; in this case the variant rule is used.
3. When applying a translation rule, any antecedent which is not an instance of the G rule, is discharged, since it is an assumption of the gap we are filling (see Proposition 1).

We would like to remark that in the translation rules $\mathbf{T}\forall$, $\mathbf{F_c}\forall$, $\mathbf{F_c}\forall\bot$, $\mathbf{F_c}\exists$ and $\mathbf{F_c}\exists\bot$ the active formula is duplicated in the gap. In fact, according to our tableau calculus, the rules $\mathbf{T}\forall$, $\mathbf{F_c}\forall$ and $\mathbf{F_c}\exists$ require duplicating the active formula. In order to maintain the correspondence between terminal nodes and gaps, as shown in Proposition 1, we must duplicate the active formula in the translation rule. But, in the proof obtained when translating a closed tableau, these assumptions may disappear since, as noted in the remark after Definition 5, unneeded assumptions are deleted by the gap closure operation.

$\dfrac{A \wedge B \qquad \dfrac{\Gamma, [A], [B]}{D}\,\mathrm{G}}{D}\,\mathbf{T}\wedge$	
$\dfrac{\dfrac{\Gamma}{D \vee A}\,\mathrm{G} \qquad \dfrac{\Gamma}{D \vee B}\,\mathrm{G}}{D \vee (A \wedge B)}\,\mathbf{F}\wedge$	$\dfrac{\neg(A \wedge B) \qquad \dfrac{\Gamma, [\neg A]}{\bot}\,\mathrm{G} \qquad \dfrac{\Gamma, [\neg B]}{\bot}\,\mathrm{G}}{D}\,\mathbf{F_c}\wedge$
$\dfrac{\dfrac{\Gamma}{A}\,\mathrm{G} \qquad \dfrac{\Gamma}{B}\,\mathrm{G}}{A \wedge B}\,\mathbf{F}\wedge\mathrm{no}\vee$	$\dfrac{\neg(A \wedge B) \qquad \dfrac{\Gamma, [\neg A]}{\bot}\,\mathrm{G} \qquad \dfrac{\Gamma, [\neg B]}{\bot}\,\mathrm{G}}{\bot}\,\mathbf{F_c}\wedge\bot$
$\dfrac{A \vee B \qquad \dfrac{\Gamma, [A]}{D}\,\mathrm{G} \qquad \dfrac{\Gamma, [B]}{D}\,\mathrm{G}}{D}\,\mathbf{T}\vee$	
$\dfrac{\neg(A \vee B) \qquad \dfrac{\Gamma, [\neg A], [\neg B]}{D}\,\mathrm{G}}{D}\,\mathbf{F_c}\vee$	
$\dfrac{\dfrac{\Gamma, [A]}{\bot}\,\mathrm{G}}{D \vee \neg A}\,\mathbf{F}\neg$	$\dfrac{\dfrac{\Gamma, [A]}{\bot}\,\mathrm{G}}{\neg A}\,\mathbf{F}\neg\mathrm{no}\vee$
$\dfrac{\neg\neg A \qquad \dfrac{\Gamma, [A]}{\bot}\,\mathrm{G}}{D}\,\mathbf{F_c}\neg$	$\dfrac{\neg\neg A \qquad \dfrac{\Gamma, [A]}{\bot}\,\mathrm{G}}{\bot}\,\mathbf{F_c}\neg\bot$
$\dfrac{\dfrac{\Gamma, [A]}{B}\,\mathrm{G}}{D \vee (A \to B)}\,\mathbf{F}\to$	$\dfrac{\dfrac{\Gamma, [A]}{B}\,\mathrm{G}}{A \to B}\,\mathbf{F}\to\mathrm{no}\vee$
$\dfrac{\neg(A \to B) \qquad \dfrac{\Gamma, [A], [\neg B]}{\bot}\,\mathrm{G}}{G}\,\mathbf{F_c}\to$	$\dfrac{\neg(A \to B) \qquad \dfrac{\Gamma, [A], [\neg B]}{\bot}\,\mathrm{G}}{\bot}\,\mathbf{F_c}\to\bot$
$\dfrac{A \to B \quad \dfrac{\Gamma}{D \vee A}\,\mathrm{G} \quad \dfrac{\Gamma, [B]}{D}\,\mathrm{G}}{D}\,\mathbf{T}\to a\neg$ where A is atomic or negated	$\dfrac{A \to B \quad \dfrac{\Gamma}{A}\,\mathrm{G} \quad \dfrac{\Gamma, [B]}{\bot}\,\mathrm{G}}{\bot}\,\mathbf{T}\to a\neg\bot$ where A is atomic or negated
$\dfrac{A \vee B \to C \qquad \dfrac{\Gamma, [A \to C], [B \to C]}{D}\,\mathrm{G}}{D}\,\mathbf{T}\to\vee$	$\dfrac{A \wedge B \to C \qquad \dfrac{\Gamma, [A \to (B \to C)]}{D}\,\mathrm{G}}{D}\,\mathbf{T}\to\wedge$

$$\dfrac{(A \to B) \to C \qquad \dfrac{\Gamma, [B \to C]}{D \vee (A \to B)}\,\mathrm{G} \qquad \dfrac{\Gamma, [C]}{D}\,\mathrm{G}}{D}\ \mathbf{T}{\to}{\to}$$

$$\dfrac{(A \to B) \to C \qquad \dfrac{\Gamma, [B \to C]}{A \to B}\,\mathrm{G} \qquad \dfrac{\Gamma, [C]}{\bot}\,\mathrm{G}}{\bot}\ \mathbf{T}{\to}{\to}\bot$$

$$\dfrac{(\forall x.\, A(x)) \to B \qquad \dfrac{\Gamma, (\forall x.\, A(x)) \to B}{D \vee (\forall x.\, A(x))}\,\mathrm{G} \qquad \dfrac{\Gamma, [B]}{D}\,\mathrm{G}}{D}\ \mathbf{T}{\to}\forall$$

$$\dfrac{(\forall x.\, A(x)) \to B \qquad \dfrac{\Gamma, (\forall x.\, A(x)) \to B}{\forall x.\, A(x)}\,\mathrm{G} \qquad \dfrac{\Gamma, [B]}{\bot}\,\mathrm{G}}{\bot}\ \mathbf{T}{\to}\forall\bot$$

$$\dfrac{(\exists x.\, A(x)) \to B \qquad \dfrac{\Gamma, [\exists x.\, A(x) \to B]}{D}\,\mathrm{G}}{D}\ \mathbf{T}{\to}\exists$$

$$\dfrac{(\exists x.\, A(x)) \to B \qquad \dfrac{\Gamma, [\exists x.\, A(x) \to B]}{\bot}\,\mathrm{G}}{\bot}\ \mathbf{T}{\to}\exists\bot$$

$$\dfrac{\forall x.\, A(x) \qquad \dfrac{\Gamma, [A(a)], \forall x.\, A(x)}{D}\,\mathrm{G}}{D}\ \mathbf{T}\forall$$

$$\dfrac{\dfrac{\Gamma}{A(p)}\,\mathrm{G}}{D \vee \forall x.\, A(x)}\ \mathbf{F}\forall$$

with p eigenvariable

$$\dfrac{\dfrac{\Gamma}{A(p)}\,\mathrm{G}}{\forall x.\, A(x)}\ \mathbf{F}\forall\mathrm{no}\vee$$

with p eigenvariable

$$\dfrac{\neg\forall x.\, A(x) \qquad \dfrac{\Gamma, \neg\forall x.\, A(x)}{A(p)}\,\mathrm{G}}{D}\ \mathbf{F_c}\forall$$

with p eigenvariable

$$\dfrac{\neg\forall x.\, A(x) \qquad \dfrac{\Gamma, \neg\forall x.\, A(x)}{A(p)}\,\mathrm{G}}{\bot}\ \mathbf{F_c}\forall\bot$$

with p eigenvariable

$$\dfrac{\exists x.\, A(x) \qquad \dfrac{\Gamma, [A(p)]}{D}\,\mathrm{G}}{D}\ \mathbf{T}\exists$$

with p eigenvariable

$$\dfrac{\neg\exists x.\, A(x) \qquad \dfrac{\Gamma, [\neg A(a)], \neg\exists x.\, A(x)}{D}\,\mathrm{G}}{D}\ \mathbf{F_c}\exists$$

$$\dfrac{\dfrac{\Gamma}{D \vee A(a)}\,\mathrm{G}}{D \vee \exists x.\, A(x)}\ \mathbf{F}\exists$$

$$\dfrac{\neg\exists x.\, A(x) \qquad \dfrac{\Gamma, [\neg A(a)], \neg\exists x.\, A(x)}{\bot}\,\mathrm{G}}{\bot}\ \mathbf{F_c}\exists\bot$$

$$\dfrac{\dfrac{\Gamma}{A(a)}\,\mathrm{G}}{\exists x.\, A(x)}\ \mathbf{F}\exists\mathrm{no}\vee$$