Formal Connected Basic Pairs

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What is a connected space?

- Following J.R. Munkres, *Topology*, Prentice Hall (2000, 2nd edition):

> *One says that a space can be "separated" if it can be broken up into two "globs" — disjoint open sets.*
> *Otherwise, one says that it is connected.*

Definition (Connected space)

Let X be a topological space.
A *separation* of X is a pair A, B of disjoint nonempty open subsets of X whose union is X.
The space X is said to be *connected* if there does not exist a separation of X.

A few comments

- Definition I is *negative*: a connected set does **not** have the property of being separated.
- Definition I implies a quantification over open subsets. And we cannot limit the quantification to elements of a basis.
- Definition I requires points. In fact, A and B must be nonempty (having at least one point) and disjoint (no points in common).
- The goal of this talk is to give a point-free definition of connected space and connected (sub)set in the framework of formal topology, specifically in Sambin's Basic Picture.

Connected subsets

Definition (Connected subset)

A subset $E \subseteq X$ is said to be *connected* in the topological space X if the corresponding subspace E is connected.

- The definition is easy and natural
- It requires the auxiliary notion of subspace.

Formal connected spaces

Definition (Connected space)

Let X be a topological space. A *separation* of X is a pair A, B of disjoint nonempty open subsets of X whose union is X.
The space X is said to be *connected* if there does not exist a separation of X.

- The first step in building a constructive version is to formulate Definition I in the language of basic pairs.
- We notice that, in classical logic, being A and B nonempty open subsets, the definition is equivalent to

$$X \subseteq A \cup B \rightarrow \exists x \in X . x \in A \cap B \ .$$

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$$X \subseteq A \cup B \rightarrow \exists x \in X . x \in A \cap B \ .$$

- Let's suppose to have the basic pair $\langle X; \Vdash; S \rangle$.
- A is an open subset if and only if $A = \mathrm{ext}\, U$ for some $U \subseteq S$. Similarly, $B = \mathrm{ext}\, V$.
- Moreover, supposing $S = \Diamond X$, i.e., $\forall a \in S . \mathrm{ext}\, a \between X$, the open subset $A = \mathrm{ext}\, U$ is nonempty if and only if U is nonempty. Similarly for V.
- The previous condition can be rephrased as, given U and V nonempty subsets of S,

$$X \subseteq \mathrm{ext}\, U \cup \mathrm{ext}\, V \rightarrow \mathrm{ext}\, U \between \mathrm{ext}\, V \ .$$

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$$X \subseteq \mathrm{ext}\, U \cup \mathrm{ext}\, V \to \mathrm{ext}\, U \between \mathrm{ext}\, V \ .$$

- Thus, supposing that $X = \mathrm{ext}\, S$, for nonempty U and V, the condition is equivalent to

$$\mathrm{ext}\, S \subseteq \mathrm{ext}(U \cup V) \to \mathrm{ext}\, U \between \mathrm{ext}\, V \ .$$

- By definition of $\Box$, the condition can be rephrased as

$$S \subseteq (\Box \circ \mathrm{ext})(U \cup V) \to \mathrm{ext}\, U \between \mathrm{ext}\, V \ .$$

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$$S \subseteq (\Box \circ \mathrm{ext})(U \cup V) \to \mathrm{ext}\, U \between \mathrm{ext}\, V \ .$$

- By definition of $\mathcal{A}$ and $\Diamond$,

$$S \subseteq \mathcal{A}(U \cup V) \to U \between (\Diamond \circ \mathrm{ext})V \ .$$

- It becomes natural to define a new operator $\mathrm{conn} = \Diamond \circ \mathrm{ext}$:

$$S \subseteq \mathcal{A}(U \cup V) \to U \between \mathrm{conn}\, V \ .$$

- It is immediate to show that the previous condition is equivalent to the symmetrical

$$S \subseteq \mathcal{A}(U \cup V) \to V \between \mathrm{conn}\, U \ .$$

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- The previous condition encodes the notion of connected space if $X = \mathrm{ext}\, S$, $S = \Diamond X$ and both U and V are nonempty.
- The first and the second assumptions are *structural* so it makes sense to single out basic pairs meeting them:

Definition (Spatial basic pair)

A basic pair $\langle X; \Vdash; S \rangle$ is *spatial* if $X = \mathrm{ext}\, S$ and $S = \Diamond X$.

- A spatial basic pair represents a space having no point outside any open subset (since $X = \mathrm{ext}\, S$), and no empty basic neighbourhood (since $S = \Diamond X$).

Formal connected spaces

- Intuitively, $\mathrm{conn}A$ is the collection of basic neighbourhoods having a point in common with A, that is, $a \in \mathrm{conn}A$ if and only if a is *directly connected* to A.
- If, by symmetry, we define $\mathrm{nocc} = \mathrm{ext} \circ \Diamond$, it is possible to describe in a slightly different way the condition of being spatial.
- In fact, it is easy to prove that the assumptions $S = \Diamond X$ and $X = \mathrm{ext}S$ are equivalent to $S = \mathrm{conn}S$ and $X = \mathrm{nocc}X$, respectively.
- Although this alternative definition has an appeal, its content is the same as the given one and no benefit is gained adopting it.

Formal connected spaces

Definition (Formal connected space)

A spatial basic pair $\langle X; \Vdash; S \rangle$ is *connected* if and only if the following condition holds for any U, V subsets of S:

$$U \neq \emptyset \wedge V \neq \emptyset \wedge S = \mathcal{A}(U \cup V) \rightarrow U \between \mathrm{conn}\, V \quad .$$

- The condition is *formal*, i.e., it lives only in the formal side of the basic pair.
- It is *reasonable* since the condition says that, when U and V are nonempty and their union covers S, then the collection of basic neighbourhoods directly connected to V has an element in common with U.

Subspaces

- Spatial basic pairs have a natural definition that closely fits our intuition.
- But they also suggest how to define a coherent notion of subspace.

Definition (Subspace)

Given a spatial basic pair $\mathbb{S} = \langle X; \Vdash; S\rangle$ and a subset $Y \subseteq X$, we say that the subspace on Y in $\mathbb{S}$ is the basic pair $\mathbb{Y} = \langle X; \Vdash_Y; S\rangle$ where $x \Vdash_Y a \equiv x \Vdash a \wedge x \in Y$.

- Evidently, $\mathrm{ext}_Y U = \mathrm{ext}\, U \cap Y$ so the open subsets of $\mathbb{Y}$ are exactly the open sets in the classical construction of the subspace topology over Y.

Subspaces

- A few calculations lead to prove that $\mathrm{conn}_Y S = \Diamond Y$, that is, the basic neighbourhoods immediately connected to S in $\mathbb{Y}$ are the basic neighbourhoods having at least one point in Y (in $\mathbb{S}$).
- Similarly, one proves that $\mathrm{nocc}_Y X = \mathrm{ext} S \cap Y$, that is, the points of X which are "glued" to some other point via the basic neighbourhoods of $\mathbb{Y}$ are exactly the points of Y lying in some basic neighbourhood of S in $\mathbb{S}$.
- Since $\mathbb{S}$ is spatial, $\mathrm{ext} S \cap Y = X \cap Y = Y$, so $\mathrm{nocc}_Y X = Y$.
- Therefore, the "domain" of the $\Vdash_Y$ relation is exactly $\mathrm{nocc}_Y X$; analogously, the domain of $\Vdash$ is $\mathrm{nocc} X$.
- Similarly, the "range" of the $\Vdash_Y$ relation is exactly $\mathrm{conn}_Y S$; analogously, the range of $\Vdash$ is $\mathrm{conn} S$.

Subspaces

- Thus, a subspace $\langle X; \Vdash_Y; S \rangle$ is thought as the "topology" on the points corresponding to the domain of $\Vdash_Y$, whose basis is composed by the extensions of the basic neighbourhoods in the range of $\Vdash_Y$.
- Since the domain of $\Vdash_Y$ is $\mathrm{nocc}_Y X = \mathrm{ext} S \cap Y = Y$ and the range of $\Vdash_Y$ is $\mathrm{conn}_Y S = \Diamond Y$, we are able to express the relevant notions of our interpretation of the subspace both inside its language and in the language of the space.

Formal connected subspaces

- To define the notion of connected subspace, we want to say that, for every pair of nonempty open subsets in the subspace, if their union is the domain of $\Vdash_Y$, then their intersection must contain at least one point.
- Since we can prove that $\mathrm{ext}_Y U = Y \cap \mathrm{ext}\, U$ and $\mathrm{ext}\, U \between Y \equiv U \between \mathrm{conn}_Y S$, which is a condition lying on the formal side of the subspace, we can define:

Definition (Formal connected subspace)

A subspace $\langle X; \Vdash_Y; S \rangle$ is *connected* if and only if the following condition holds for any U, V subsets of S:

$$U \between \mathrm{conn}_Y S \wedge V \between \mathrm{conn}_Y S \wedge S = \mathcal{A}_Y(U \cup V) \rightarrow U \between \mathrm{conn}_Y V \quad .$$

Formal connected spaces

- The previous definition suggests to abstract and to generalise:

Definition (Formal connected basic pair)

A basic pair $\langle X; \Vdash; S\rangle$ is *connected* if and only if the following condition holds for any U, V subsets of S:

$$U \between \mathrm{conn} S \wedge V \between \mathrm{conn} S \wedge S = \mathcal{A}(U \cup V) \rightarrow U \between \mathrm{conn} V \ .$$

- Evidently, this definition applies to general basic pairs, not just spatial basic pairs or subspaces.

Formal connected spaces

Definition (Formal connected basic pair)

A basic pair $\langle X; \Vdash; S\rangle$ is *connected* if and only if the following condition holds for any U, V subsets of S:

$$U \between \mathrm{conn} S \wedge V \between \mathrm{conn} S \wedge S = \mathcal{A}(U \cup V) \to U \between \mathrm{conn} V \quad .$$

- (**interpretation**) We stipulate that the "points" are the elements in the domain of $\Vdash$, while the basis of the "topology" is denoted by the range of $\Vdash$. Hence the collection of points is $\mathrm{ext} S$ while the basis is $\Diamond X$.

Formal connected spaces

Definition (Formal connected basic pair)

A basic pair $\langle X; \Vdash; S \rangle$ is *connected* if and only if the following condition holds for any U, V subsets of S:

$$U \between \mathrm{conn}\, S \wedge V \between \mathrm{conn}\, S \wedge S = \mathcal{A}(U \cup V) \to U \between \mathrm{conn}\, V \ .$$

- Since $U \between \mathrm{conn}\, S \equiv \mathrm{ext}\, U \between \mathrm{ext}\, S$, U denotes a open subset having at least one point, i.e., it is “nonempty”.
- Since $S = \mathcal{A}(U \cup V) \equiv \mathrm{ext}\, S = \mathrm{ext}\, U \cup \mathrm{ext}\, V$, the union of the points in U and V is the collection of all points.
- Finally, $U \between \mathrm{conn}\, V \equiv \mathrm{ext}\, U \between \mathrm{ext}\, V$, so there is a point lying in the open subsets denoted by U and V.

Conclusions

- We have shown a point-free definition of connected space and connected subset valid for generic basic pairs.
- The definition, under the correct interpretation, is natural and intuitive (well, if one knows the Basic Picture).
- As a “subproduct”, we defined subspaces as a special construction over basic pairs.

- Most of the elementar results about connected spaces can be easily proved using the given definition.

Future directions

- Path-connection is easy if we define paths as functions.
- But the Basic Picture provides a clean and elegant notion of continuous relation...
- ... which naturally suggests to use “relational paths”.
- We are exploring relational paths and the topic is NOT immediate.
- But this is for another talk, sometime in the future...

Thanks

- First and most important to **Giovanni Sambin** since, without the Basic Picture, nothing in this talk would make sense.
- **Brunella Gerla**, my colleague who shares the efforts behind this work and owes its merits.
- **Peter Schuster**, who kindly invited me in this workshop and who constantly reviews my ideas (and my mistakes).

Lemma

$\mathrm{conn} S = \Diamond X$.

Proof.

$a \in \mathrm{conn} S \equiv a \in (\Diamond \circ \mathrm{ext}) S \equiv \exists x \in X . a \in \Diamond x \wedge x \in \mathrm{ext} S \equiv \exists x \in X . a \in \Diamond x \wedge \Diamond x \between S$.

But $a \in S$, so $a \in \mathrm{conn} S \equiv \exists x \in X . a \in \Diamond x \equiv \exists x \in X . x \in \mathrm{ext} a \equiv \mathrm{ext} a \between X \equiv a \in \Diamond X$. □

Slide 16

Lemma

$\mathrm{nocc}S = \mathrm{ext}S.$

Proof.

$x \in \mathrm{nocc}X \equiv x \in (\mathrm{ext} \circ \Diamond)X \equiv \exists a \in S.x \in \mathrm{ext}a \wedge a \in \Diamond S \equiv \exists a \in S.x \in \mathrm{ext}a \wedge \mathrm{ext}a \between X$.
But $x \in X$, so $x \in \mathrm{nocc}X \equiv \exists a \in S.x \in \mathrm{ext}a \equiv \exists a \in S.a \in \Diamond x \equiv \Diamond x \between S \equiv x \in \mathrm{ext}S$. □

Slide 18 and 21

Lemma

$\mathrm{ext}_Y U = Y \cap \mathrm{ext}\, U$.

Proof.

$x \in \mathrm{ext}_Y U \equiv \exists a \in U. x \in \mathrm{ext}_Y a \equiv \exists a \in U. x \Vdash_Y a \equiv \exists a \in U. x \in Y \wedge x \Vdash a \equiv x \in Y \wedge \exists a \in U. x \Vdash a \equiv x \in Y \wedge x \in \mathrm{ext}\, U \equiv x \in Y \cap \mathrm{ext}\, U$. □

Slide 19

Lemma

$\mathrm{conn}_Y S = \Diamond Y$.

Proof.

$a \in \mathrm{conn}_Y S \equiv \exists x \in X.x \Vdash a \wedge x \in Y \wedge x \in \mathrm{ext}_Y S \equiv \exists x \in Y.x \in \mathrm{ext}\, a \wedge x \in \mathrm{ext}\, S$.

But $a \in S$, so

$a \in \mathrm{conn}_Y S \equiv \exists x \in Y.x \in \mathrm{ext}\, a \equiv \mathrm{ext}\, a \between Y \equiv a \in \Diamond Y$. □

Slide 19

Lemma

$\mathrm{nocc}_Y X = Y \cap \mathrm{ext}\, S.$

Proof.

$x \in \mathrm{nocc}_Y X \equiv \exists a \in S. x \Vdash a \wedge x \in Y \wedge a \in \Diamond_Y X \equiv \exists a \in S. x \Vdash a \wedge x \in Y \wedge (\exists z \in X. z \Vdash a \wedge z \in X)$.
But $x \in X$, so $x \in \mathrm{nocc}_Y X \equiv \exists a \in S. x \Vdash a \wedge x \in Y \equiv x \in Y \wedge x \in \mathrm{ext}\, S \wedge x \in Y \cap \mathrm{ext}\, S$. □

Lemma

$U \between \mathrm{conn}_Y S \equiv \mathrm{ext}\, U \between Y$.

Proof.

$U \between \mathrm{conn}_Y S \equiv \mathrm{ext}_Y U \between \mathrm{ext}_Y S \equiv \exists x \in X. x \in \mathrm{ext}_Y U \wedge x \in \mathrm{ext}_Y S \equiv \exists x \in X.(\exists a \in U. x \Vdash a \wedge x \in Y) \wedge (\exists b \in S. x \Vdash b \wedge x \in Y)$.

But $U \subseteq S$, so $U \between \mathrm{conn}_Y S \equiv \exists x \in X, a \in U. x \Vdash a \wedge x \in Y \equiv \exists x \in X. x \in \mathrm{ext}\, U \wedge x \in Y \equiv \mathrm{ext}\, U \between Y$. □